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Advances in quantization: quantum tensors, explicit star-products, and restriction to irreducible leaves[1]

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Abstract: The notion of quantum vector fields and quantum tensors is investigated and applied for explicit calculation of the Weyl star-product over Poisson manifolds, as well as, of the Wick star-product and the reproducing measure over (pseudo) Kählerian manifolds. Quantum phase space structure over Poisson manifolds is discussed and used for the derivation of explicit formulas for irreducible representations of the star-product algebra. The quantum restriction of the star-product onto irreducible leaves is described. Some interesting geometrical constructions (for instance, a series of higher order differential invariants over Kählerian manifolds) are obtained as corollaries of quantum calculations.

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1. Introduction

Since the times of discovery of the Hamilton equations the possibility to move not only by inertia could be interpreted as a response to a noncommutative structure given "by God" on the phase manifold. This structure is represented by a skew product on cotangent vectors, i.e., by a Poisson tensor over the manifold.

In fact, this tensor was already the first sign of a "quantum," and the first object really belonging to the quantization area. The classical commutative algebra of functions over the manifold was first filled by a noncommutative spirit via the Poisson brackets.

The Lie-algebraic framework is now widely accepted by geometers (brackets of functions, of vector fields, of tensors, of forms, etc.), but associative functional algebras and calculus of noncommuting operators are still considered as alien. This orthodox geometric view will possibly begin to change thanks to the coherent states theory [5, 6, 8, 14, 57], to the deformation quantization [29, 3, 4, 20, 18, 19, 27, 28, 33, 60, 72, 73, 76–78, 88, 89, 94, 97], to the asymptotic (groupoid) quantization [42, 43, 46–49, 51–54, 92, 93, 96, 99], and to Connes' noncom-

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mutative geometry [17, 23, 24]. Geometric discoveries succeeded to make by purely quantum instruments demonstrate that "quantization" is going to reconcile forever the differential geometry and the noncommutative analysis.

What we now call "quantization" is a noncommutative and associative product $\star$ given on a set of functions over a manifold. A manifold $\mathcal{M}$ with such an associative $\star$-product is called a *quantum manifold.*

In this associative framework some important "infinitesimal" questions appear immediately. What is a quantum analog of the Poisson tensor? What are quantum analogs of general vector fields, tensors, Schouten brackets? How to reconstruct the $\star$-product over $\mathcal{M}$ explicitly via the infinitesimal quantum $\star$-tensor? What are relations between the quantum $\star$-tensor on the manifold $\mathcal{M}$ and the quantum $\star$-tensor on quantum irreducible leaves in $\mathcal{M}$, or on the quantum phase space over $\mathcal{M}$?

In this paper we describe the following constructions:
- a quantum version of the classical Schouten–Lichnerowicz tensor complex [64],
- a nonlinear analog of the construction of the Weyl $\star$-product developed in the case of linear Poisson tensors by Berezin [10],
- an explicit construction of the Wick star-product over Kähler manifolds,
- a quantum restriction of the star-product onto irreducible leaves,
- a quantum restriction of the quantum reduction mappings.

The first topic of the paper is devoted to the derivation of some *quantum version of the classical Jacobi tensor identity*. This quantum identity is equivalent to the associativity condition for the $\star$-product. A quantum tensor satisfying this identity we call a *quantum $\star$-tensor*. The identity itself looks like a highly nonlinear generalization of the Lie relations for structure constants of a Lie algebra (the linear case), as well as of the quantum Yang–Baxter equation (the quadratic case).

At the same time we show an *explicit procedure for calculating the Weyl $\star$-product* in all orders of the deformation parameter, starting from a given quantum $\star$-tensor. Here we introduce and apply a notion of *quantum vector fields* and *quantum tensors* which is different from that usually exploited.

These results allow us to complete the previous calculations concerning the *extension* of a quantum manifold to a quantum phase space of doubled dimension with a nondegenerate $\star$-product (i.e., having the trivial center). In the classical limit this extended space is just the phase space over the Poisson manifold in the sense of [42, 51, 52], or the symplectic realization in the sense of [90], with a pair of "reduction" Poisson mappings, whose components mutually commute. On the quantum level the Poisson mappings should be replaced by a pair of mutually commuting *reduction homomorphisms* into the function algebra over the quantum phase space. Now, thanks to the explicit construction of the Weyl $\star$-product, we are provided by a procedure for calculating the reduction homomorphisms only in terms of the quantum $\star$-tensor. So, we obtain an answer to the quantum version of the well-known question about classical Poisson tensors due to Sophus Lie [65].

In the case of a nondegenerate Poisson tensor, i.e., in the symplectic case, it is natural to try to avoid quantum Jacobi conditions for the quantum tensor and just use the symplectic 2-form (or its deformations) as a basic object generating the star-product. An elegant geometric procedure for this was discovered in the work by Fedosov [27, 28] (which strongly exploited the notion of symplectic connection), and also in [72], for the case of Weyl products. For the Wick–Klauder–

Berezin case the same was made in [11]. In all these cases formulas for star-products are not given in the explicit form and still need a clarification. We show below, in a very simple and explicit way, how to derive the *Wick product $*$ over an arbitrary (pseudo) Kähler manifold* starting from the Kählerian 2-form and exploiting the idea of quantum vector fields (in its complex version), as well the creation-annihilation operators. Parallel with the derivation of the product $*$ we also obtain a *simple asymptotic expansion for the quantum $*$-tensor*, which automatically satisfies the quantum Jacobi conditions, and an expansion for the relevant *reproducing measure*.

An interesting geometrical consequence of this quantum derivation is a series of *higher order differential invariants*, as well as a series of closed 1-forms, which we explicitly calculate over an arbitrary (pseudo) Kähler manifold.

The second topic of the paper concerns with an explicit construction of the *irreducible representations* of the star-product algebra. For this purpose, we use the quantum extension procedure (in the complex version), a notion of *vacuum submanifold*, *creation/annihilation submanifolds*, and the Wick dequantization.

This construction allows us to justify the above manifesto—

"*Quantization commutes with restriction onto irreducible leaves*"

with the essential comment that the "restriction" should be understood in an appropriate quantum sense. Guillemin's brilliant lecture at the Newton Mathematical Institute (Cambridge, 1994) began with the declaration: "All my life I know that quantization commutes with reduction." So, our above manifestation is just an imitation of this life principle of people in the geometric quantization area; see details in [13, 34, 32, 30, 62, 67, 87], as well as in Kirwan's paper in this volume.

We show that the set of functions, whose quantum restriction on a quantum irreducible leaf Ω equals zero, is a two-side ideal in the function algebra over the entire quantum manifold $\mathcal{M}$. The quotient algebra by this ideal is isomorphic to the algebra of functions over the quantum leaf. The isomorphism is derived explicitly. As a result, the quantum $*$-tensor determining the $*$-product over $\mathcal{M}$ can be restricted to a quantum $*$-tensor over the leaf Ω. Such a procedure of *quantum restriction* gives a positive answer to the known problem about "consistent quantization" of Poisson manifolds and their symplectic leaves. The discussion and some very interesting negative results around this problem see in [15, 19, 80, 95]. In the framework of geometric quantization the pullback to symplectic leaves was considered in [86].

In the classical limit the word "quantum" has to be omitted, and irreducible leaves are just symplectic leaves in the Poisson manifold. But in the quantum case, in general, there are corrections to the classical limit values of all objects considered. The above manifestation—quantization commutes with restriction—does not hold without these corrections. It seems very probable that quantum corrections appearing in our definition of the quantum restriction are similar (in a particular case) to those observed by Kostant and Sternberg [62, 25] in the geometric quantization of constraint systems.

Note that the quantum restriction of functions from $\mathcal{M}$ onto Ω is generated, in fact, by a *quantum embedding* of the manifold Ω into $\mathcal{M}$, and quantum leaves can be considered as irreducible quantum submanifolds in the quantum manifold.

Actually, we also construct a pair of quantum embeddings of the phase space over Ω into the (complexified) phase space over $\mathcal{M}$. The images of these embeddings are "creation" and "annihilation" symplectic submanifolds. The embeddings are generated by a *quantum Kählerian potential* obtained as the solution of a certain multi-time Cauchy problem for quantum Hamilton–Jacobi equations with quantum-commuting Hamilton functions.

On this basis we finally prove in conclusion of the paper that

"Quantum reduction homomorphisms commute with quantum restriction."

Here the "restriction" is understood either as the restriction onto quantum irreducible leaves or as the restriction onto quantum creation and annihilation submanifolds, depending on what part of the commutative diagram is considered (see formulas (14.10) below).

2. Preliminaries

Let $\mathcal{M}^n$ be a manifold with a formal $\star$-product given by a power series in $\hbar$. So, the space $C^\infty(\mathcal{M}, [\hbar])$ of formal power series in $\hbar$ with smooth coefficients is an associative algebra with the unity 1 [3, 4]. The product $\star$ is called *local* [73] if at each point the function $f \star g$ depends only on the germs of f and g at this point. In this case:

- over each local chart $\mathcal{U} \subset \mathcal{M}$ with coordinate mapping $\xi : \mathcal{U} \to \mathbb{R}^n$ there is a local $\star$-product in $C^\infty(\xi(\mathcal{U}), [\hbar])$,
- for each pair of intersecting charts $(\mathcal{U}, \xi)$ and $(\mathcal{U}', \xi')$, there is a patching morphism of algebras
$$C^\infty\big(\xi(\mathcal{U} \cap \mathcal{U}'), [\hbar]\big) \to C^\infty\big(\xi'(\mathcal{U} \cap \mathcal{U}'), [\hbar]\big) \tag{2.1}$$
with the usual cocyclic condition on intersections of triples of charts,
- the algebra $C^\infty(\mathcal{M}, [\hbar])$ is isomorphic to the algebra of sections of the sheaf of algebras over local charts.

In what follows, we shall refer to these sections as functions over $\mathcal{M}$.

Below, for simplicity, we suppose that $\mathcal{M}$ is an analytic manifold, all functions are from the class C^ω, and the $\star$-product is of C^ω-type.

The local $\star$-product on $\mathcal{M}$ is called the *Weyl product* if, in each chart, in some local coordinates $\xi = (\xi^1, \ldots, \xi^n)$ the following identities hold (the Weyl symmetrization rule):

$$\xi^\alpha = \langle \xi \rangle^\alpha, \qquad \forall\, \alpha = (\alpha_1, \ldots, \alpha_n).$$

Here, on the left we denote $\xi^\alpha = (\xi^1)^{\alpha_1} \ldots (\xi^n)^{\alpha_n}$ and on the right

$$\langle \xi \rangle^\alpha \stackrel{\text{def}}{=} \frac{\alpha!}{|\alpha|!} \sum_{j \in \mathcal{A}_\alpha} \xi^{j_1} \star \cdots \star \xi^{j_{|\alpha|}}, \tag{2.2}$$

where the sum is taken over the set $\mathcal{A}_\alpha$ of all mappings $j\colon (1, \ldots, |\alpha|) \to (1, \ldots, n)$ that take each value k exactly α_k times.

In this case the patching morphisms (2.1) are given by

$$f \to f', \qquad f\langle \varphi(\xi') \rangle = f'(\xi'), \tag{2.3}$$

where $\xi = \varphi(\xi')$ are changes of local coordinates (on an intersection of charts). On the left in (2.3), the symmetrization (2.2) is applied to noncommuting elements $\varphi^1(\xi'), \ldots, \varphi^n(\xi')$ of the $\star$-product algebra over $\xi'(\mathcal{U}')$. In the classical limit $\hbar = 0$, the mappings φ are the gluing mappings of the manifold $\mathcal{M}$.

The quantum manifold $\mathcal{M}$ with such a $\star$-product is called a *Weyl manifold* (compare with [72, 73, 71] in the nondegenerate, i.e., symplectic case).

In several sections that follow, we describe Weyl manifolds, using infinitesimal quantum geometric objects like quantum vector fields, quantum tensors, etc.

First of all, let us introduce a notation for the quantum brackets of functions

$$[f, g]_\hbar \stackrel{\text{def}}{=} \frac{i}{\hbar}(f \star g - g \star f).$$

Now in each local chart with Weyl local coordinates $\xi^1, \dots, \xi^n$ we can define the set of functions

$$K^{\ell s}(\xi) = [\xi^\ell, \xi^s]_\hbar, \qquad \ell, s = 1, \dots, n, \tag{2.4}$$

and call this set the $\star$-*tensor* on $\mathcal{M}$. We stress that it is not a tensor in the usual sense (see below).

Of course, the classical limit of the $\star$-tensor

$$P^{\ell s} = \lim_{\hbar \to 0} K^{\ell s}$$

is the classical Poisson tensor on $\mathcal{M}$. In the expansion

$$K^{\ell s} = P^{\ell s} + \hbar^2 P^{\ell s}_{(1)} + \hbar^4 P^{\ell s}_{(2)} + \cdots, \tag{2.5}$$

all higher coefficients $P^{\ell s}_{(1)}, P^{\ell s}_{(2)}, \dots$ are called [3, 88] *quantum corrections* to the Poisson tensor. These corrections carry the information about the associativity of the $\star$-product. The question is: *what are the conditions for $K^{\ell s}$ that are equivalent to the associativity of the $\star$-product?*

The conditions for $P^{\ell s}$ are obvious: the classical Jacobi identities. The conditions for the first quantum corrections $P^{\ell s}_{(1)}$ where described in [54]. Now we consider this question systematically for all higher orders of $\hbar$.

We also obtain an explicit formula for the Weyl $\star$-product via a given $\star$-tensor K. In first orders in $\hbar$ (up to $O(\hbar^4)$), these formulas were derived in [54]. Now we describe the procedure in all higher orders.

3. Quantum vector fields

Let $a = (a^1, \dots, a^n)$ be a vector-function in a local chart. For each polynomial f, we can define a Weyl-symmetrized function in the noncommuting variables $\xi^1 + \varepsilon a^1(\xi), \dots, \xi^n + \varepsilon a^n(\xi)$, by using the rule (2.2). Denote this function by $f\langle \xi + \varepsilon a(\xi)\rangle$ and expand it by ε:

$$f\langle \xi + \varepsilon a(\xi)\rangle = f(\xi) + \varepsilon a^\frown(f)(\xi) + O(\varepsilon^2). \tag{3.1}$$

We denote the coefficient at the first order in ε by $a^\frown(f)(\xi)$. So, one obtains a linear operator $a^\frown$ on the space of functions, which will be called the *quantum vector field*.

The functions a^j will be called *components* of the quantum vector field over the given local chart. The rule of transformation of components from one chart to another follows from (3.1); see below (7.4), (7.5).

Now let us derive quantum vector fields explicitly via the $\star$-product.

Denote by $\mathrm{ad}_\hbar = (\mathrm{ad}_\hbar^1, \dots, \mathrm{ad}_\hbar^n)$ the vector-operator whose components act as brackets

$$\mathrm{ad}_\hbar^j f \stackrel{\text{def}}{=} [\xi^j, f]_\hbar.$$

Also consider the vector-operator $d = (d_1, \dots, d_n)$, where $d_j = \partial/\partial\xi^j$.

Let us introduce the following *"lunar" product* of functions

$$g \odot f \stackrel{\text{def}}{=} g\, \mathbf{p}\big(-i\hbar \underset{\leftarrow}{\mathrm{ad}}_\hbar \star \underset{\rightarrow}{d}\big) f. \tag{3.2}$$

Here $\mathbf{p}(x) = (e^x - 1)/x$; the left or right arrow indicates on which a multiplier g or f the given operator acts.

So, in (3.2) the operators $\mathrm{ad}_\hbar$ act on g, the operators d act on f, and then the $\star$-product is evaluated. Expanding the functions $\mathbf{p}$ in (3.2) in the power series, we obtain the following formal power series in $\hbar$ for the lunar product:

$$g \odot f = \sum_{m=0}^{\infty} \frac{(-i\hbar)^m}{m+1} \sum_{|\alpha|=m} \frac{1}{\alpha!} (\mathrm{ad}_\hbar)^\alpha g \star d^\alpha f. \tag{3.2a}$$

This product is not associative, but very useful.

Let us also introduce the following multi-component generalization of the lunar product:

$$\big((g_1 \vee \cdots \vee g_m) \odot f\big)(\xi) \stackrel{\text{def}}{=} \langle g_1 \dots g_m f(\xi^1, \dots, \xi^n)\rangle. \tag{3.2b}$$

Here on the right-hand side we take the symmetrization (2.2) of noncommuting elements $g_1(\xi) \dots g_m(\xi), \xi^1, \dots, \xi^n$ in the $\star$-product algebra.

Theorem 3.1. (i) *The multi-component lunar product defined by* (3.2b) *in the case* $m = 1$ *coincides with the lunar product defined by* (3.2).

(ii) *Quantum vector fields can be expressed via the lunar product as follows*:

$$a^\frown = a \odot d. \tag{3.3}$$

(iii) *Quantum vector fields act on the lunar product as follows*:

$$a^\frown(g \odot f) = a^\frown(g) \odot f + (g \vee a^s) \odot d_s f. \tag{3.4}$$

Here and everywhere below the usual rule of summation over twice repeated (up and down) indices is applied.

(iv) *The space of quantum vector fields is a Lie algebra with respect to the commutator*

$$[a^\frown, b^\frown] = [a, b]_\hbar^\frown. \tag{3.5}$$

On the right the brackets $[\cdot\,,\,\cdot]_\hbar$ *are defined by*

$$[a, b]_\hbar^s \stackrel{\text{def}}{=} a^\frown(b^s) - b^\frown(a^s), \qquad s = 1, \dots, n. \tag{3.6}$$

Proof. (i) There is the following general formula in the Weyl calculus of noncommuting variables [41]

$$\langle S^j f(T)\rangle = \overset{2}{S}{}^j \int_0^1 f(\tau \overset{3}{T} + (1-\tau)\overset{1}{T})\, d\tau.$$

Here $T = (T^1, \dots, T^n)$ and $S = (S^1, \dots, S^n)$ are sets of elements of a noncommutative algebra, f is a polynomial, the angular brackets $\langle \dots \rangle$ mean the Weyl symmetrization (2.2), the numbers 1, 2, and 3 over the letters indicate the order of disposition of Weyl symmetrized sets of elements: the set with number 1 is placed to the extreme right position, etc. (This is Maslov's notation with overlined numbers.)

Thus, from definition (3.2b) we derive for $m = 1$:

$$(g \odot f)(\xi) = \int_0^1 d\tau\, f\langle \overset{1}{\xi} - i\hbar\tau\, \mathrm{ad}_\hbar \rangle g(\overset{2}{\xi}).$$

Here the operators $\mathrm{ad}_\hbar^j$ act only on the function g. Expand the right-hand side in the power series in $\hbar$:

$$g \odot f = \sum_{m=0}^{\infty} (-i\hbar)^m \left(\int_0^1 \tau^m\, d\tau \right) \sum_{|\alpha|=m} \frac{1}{\alpha!} \big(\langle \mathrm{ad}_\hbar \rangle^\alpha g \big) \star d^\alpha f.$$

Comparing with (3.2a), we see that the definition (3.2b) for $m = 1$ is the same as (3.2).

(ii) Again, in general algebraic notation the derivative by a parameter ε of $f(T + \varepsilon S)$ can be written as follows:

$$\frac{d}{d\varepsilon} f\langle T + \varepsilon S \rangle \Big|_{\varepsilon=0} = \langle S^j\, d_j f(T) \rangle.$$

So, in view of definition (3.1), $a^\frown(f)(\xi) = \langle a^j\, d_j f(\xi) \rangle$, and formula (3.3) follows from statement (i).

(iii) By definition (3.1), we have

$$a^\frown(g \odot f) = \frac{d}{d\varepsilon} \langle g_\varepsilon f\langle \xi + \varepsilon a \rangle \rangle \Big|_{\varepsilon=0},$$

where $g_\varepsilon = g\langle \xi + \varepsilon a \rangle = g(\xi) + \varepsilon a^\frown(g) + O(\varepsilon^2)$. Thus,

$$\begin{aligned} a^\frown(g \odot f) &= \frac{d}{d\varepsilon} \big(\langle g f\langle \xi + \varepsilon a \rangle \rangle + \varepsilon \langle a^\frown(g) f(\xi) \rangle \big) \Big|_{\varepsilon=0} \\ &= (g \vee a^s) \odot d_s f + a^\frown(g) \odot f. \end{aligned}$$

(iv) In view of (3.3) we have

$$\hat{a}\hat{b}(f) = \hat{a}(b^s \odot d_s f), \qquad \forall f.$$

Then, using (3.4) we obtain:

$$\hat{a}\hat{b} = \hat{a}(b^s) \odot d_s + (b^s \vee a^\ell) \odot d_s d_\ell. \tag{3.7}$$

And so

$$a^\frown b^\frown(f) - b^\frown a^\frown(f) = \hat{a}(b^s) \odot d_s f - \hat{b}(a^s) \odot d_s f = [a, b]_\hbar^s \odot d_s f.$$

The proof is complete. □

4. Quantum Euler–Poisson vector fields

Over each quantum manifold, there are operators of special type: inner derivations (for which the standard Leibniz rule holds).

Let us fix some function f. The *inner derivation* is given by the mapping $g \to [f, g]_\hbar$.

Theorem 4.1. (i) *Inner derivations of the $\star$-product are quantum vector fields. Namely,*

$$[f, g]_\hbar = \mathrm{ad}_\hbar(f)\hat{\ } g = -\mathrm{ad}_\hbar(g)\hat{\ } f, \tag{4.1}$$

where the components of the quantum field $\mathrm{ad}_\hbar(f)$ *are defined by*

$$\mathrm{ad}_\hbar(f)^j \overset{\text{def}}{=} -\mathrm{ad}_\hbar^j(f) = K^{si} \odot d_s f, \qquad j = 1, \dots, n. \tag{4.2}$$

In particular,

$$\mathrm{ad}_\hbar(\xi^j)\hat{\ } = \mathrm{ad}_\hbar^j, \qquad \mathrm{ad}_\hbar(\xi^j)^s = K^{js}. \tag{4.3}$$

(ii) *The formal solution of the quantum Liouville (Heisenberg) equation*

$$\frac{dF}{dt} = [H, F]_\hbar, \qquad F\Big|_{t=0} = f, \tag{4.4}$$

can be represented in the form

$$F = f\langle \Xi \rangle, \tag{4.5}$$

where $\Xi = (\Xi^1, \dots, \Xi^n)$ *is the solution of the quantum Euler–Poisson equations*

$$\frac{d\Xi}{dt} = \mathrm{ad}_\hbar(H)\langle \Xi \rangle, \qquad \Xi\Big|_{t=0} = \xi. \tag{4.6}$$

Proof. In the Weyl calculus [41] there is the general commutation formula

$$[S, g(T)] = \langle [S, T^j] d_j g(T) \rangle.$$

So,

$$[f, g]_\hbar = \frac{i}{\hbar} \langle (f \star \xi^j - \xi^j \star f) d_j g \rangle = [f, \xi^j]_\hbar \odot d_j g,$$

and

$$[f, \xi^j]_\hbar = [\xi^s, \xi^j]_\hbar \odot d_s f.$$

Hence, we have proved (4.1), (4.2). Formula (4.3) follows from (4.2). Statement (ii) is a standard consequence from (4.1) $\square$

The classical limit of (4.6) at $\hbar = 0$ looks like

$$\frac{d\Xi_0}{dt} = -(PdH)(\Xi_0), \qquad \Xi_0\Big|_{t=0} = \xi.$$

The solution $\Xi_0(t, \xi)$ is a trajectory of the Euler–Poisson vector field $\mathrm{ad}(H) \overset{\text{def}}{=} -PdH \cdot d$ on the Poisson manifold $\mathcal{M}$. The quantum version (4.6) concerns the quantum vector field $\mathrm{ad}_\hbar(H)\hat{\ } = -(K \odot dH) \odot d$. We call it the *quantum Euler–Poisson vector field.*

5. Quantum version of the Lichnerowicz–Schouten complex

Denote by $V_\hbar^0(\mathcal{M})$ the space of quantum tensors of rank 0, i.e., of functions. Denote by $V_\hbar^1(\mathcal{M})$ the space of quantum tensors of rank 1, i.e., of quantum vector fields. The quantum (skew symmetric) tensor of rank r is an element of the exterior product of r copies $V_\hbar^r(\mathcal{M}) \stackrel{\text{def}}{=} V_\hbar^1(\mathcal{M}) \wedge \cdots \wedge V_\hbar^1(\mathcal{M})$. Elements of $V_\hbar^r(\mathcal{M})$ can be represented in the form

$$A = \frac{1}{r!} A^{j_r \ldots j_1} \odot d_{j_1} \wedge \cdots \wedge d_{j_r},$$

where the set of functions $A^{j_1 \ldots j_r}$ is skew-symmetric in the multi-index j. Such tensors are identified with quantum r-vector fields as follows:

$$A(f_1, \ldots, f_r) \stackrel{\text{def}}{=} \frac{1}{r!}\big((A^{j_1 \ldots j_1} \odot d_{j_1} f_1) \odot \ldots\big) \odot d_{j_r} f_r.$$

The quantum Schouten brackets

$$[\![\cdot\,, \cdot]\!]_\hbar \colon V_\hbar^r \times V_\hbar^m \to V_\hbar^{r+m-1}, \qquad r + m \geqslant 1.$$

can be defined by analogy with the commutative case [63]

$$\begin{aligned} &[\![u_1 \wedge \cdots \wedge u_k\,,\ v_1 \wedge \cdots \wedge v_l]\!]_\hbar \\ &= \sum_{i,j} (-1)^{i+j} [u_j, v_i]_\hbar \wedge v_1 \wedge \cdots \wedge \check{v}_i \wedge \cdots \wedge v_m \wedge \cdots \wedge u_1 \wedge \cdots \wedge \check{u}_j \wedge \cdots \wedge u_r, \end{aligned}$$

where u_j and v_i are quantum vector fields (elements from $V_\hbar^1(\mathcal{M})$), and the check-sign denotes the omission.

The explicit formula for the *quantum Schouten brackets of two quantum tensors* A and B is the following

$$\begin{aligned} [\![A, B]\!]_\hbar^{j_{k+\ell-1} \ldots j_1} = \sum_{i \sim j} \bigg(&\frac{1}{\ell!\,(k-1)!} A^{i_{\ell+k-1} \ldots i_{\ell+1} s} \odot d_s B^{i_\ell \ldots i_1} \\ &+ \frac{(-1)^{\ell k + \ell + k}}{(\ell-1)!\,k!} B^{i_{k+\ell-1} \ldots i_{k+1} s} \odot d_s A^{i_k \ldots i_1} \bigg) (-1)^{\sigma(i)}. \end{aligned} \tag{5.1}$$

Here $k = \operatorname{rank} A$, $\ell = \operatorname{rank} B$, the sum $\sum_{i \sim j}$ is taken over all permutations i of the multi-index $j = (j_{k+\ell-1}, \ldots, j_1)$, and $\sigma(i)$ denotes the sign (the parity) of the permutation i.

For small ranks the list of brackets looks as follows:

$$[\![a, f]\!]_\hbar = a\hat{\ }(f) = a^s \odot d_s f$$

(brackets between rank 1 and rank 0, the result is of rank 0),

$$[\![a, b]\!]_\hbar^j = [a, b]_\hbar^j = a^s \odot d_s b^j - b^s \odot d_s a^j$$

(brackets between rank 1 and rank 1, the result is of rank 1),

$$[\![A, f]\!]_\hbar^j = A^{js} \odot d_s f$$

(brackets between rank 2 and rank 0, the result is of rank 1),

$$[\![A, a]\!]_\hbar^{j_2 j_1} = A^{j_2 s} \odot d_s a^{j_1} - A^{j_1 s} \odot d_s a^{j_2} - a^s \odot d_s A^{j_2 j_1}$$

(brackets between rank 2 and rank 1, the result is of rank 2),

$$[\![A, B]\!]_\hbar^{j_3 j_2 j_1} = \underset{(j_1, j_2, j_3)}{\mathfrak{S}} \left(A^{j_3 s} \odot d_s B^{j_2 j_1} + B^{j_3 s} \odot d_s A^{j_2 j_1}\right)$$

(brackets between rank 2 and rank 2, the result is of rank 3).

In the last formula the sum $\mathfrak{S}_{(j_1,j_2,j_3)}$ is taken over all cyclic permutations of the triple (j_1, j_2, j_3).

Note that quantum tensors over $\mathcal{M}$ form a graded Lie algebra (a superalgebra) with respect to brackets (5.1).

Let us now look at definition (2.4). From the Jacobi identities for commutator we obtain the following result.

Theorem 5.1. (i) *The* $\star$*-tensor* K (2.4) *is an element of* $V_\hbar^2(\mathcal{M})$ *such that*

$$[\![K, K]\!]_\hbar = 0,$$

or in other words,

$$\mathop{\mathfrak{S}}_{(j_1,j_2,j_3)} K^{j_3 s} \odot d_s K^{j_2 j_1} = 0, \tag{5.2}$$

where $\odot$ *is the lunar product* (3.2).

(ii) *The operation* $A \to [\![K, A]\!]_\hbar$ *can be understood as the differential in the quantum tensor complex* $\{V_\hbar^r(\mathcal{M}) \mid r = 0, 1, 2, \dots\}$.

Example 5.1. The quantum Euler–Poisson vector fields $\mathrm{ad}_\hbar(f)$ are coboundaries in this complex, since $\mathrm{ad}_\hbar(f) = -[\![K, f]\!]_\hbar$.

Theorem 5.1 represents the quantum analog of the Lichnerowicz description of Poisson manifolds via the classical Schouten brackets [64], see also in [56, 61, 66].

Now we derive the expansion for the quantum brackets $[\![\cdot,\cdot]\!]_\hbar$ in power series in $\hbar$, and hence, represent equality (5.2) as a system of equations for the quantum corrections (2.5).

6. Explicit power series for the Weyl product

There is the following key multiplication formula in the Weyl calculus [41]:

$$T^s f\langle T\rangle = \sum_{m=0}^{\infty} \frac{b_m}{m!} \langle [T^{j_1}, \dots [T^{j_m}, T^s] \dots] d_{j_1} \dots d_{j_m} f(T)\rangle. \tag{6.1}$$

Here $T = (T^1, \dots, T^n)$ is a set of noncommuting elements; $s \in (1, \dots, n)$; b_m are Bernoulli numbers; the angular brackets denote the symmetrization (2.2). The infinite sum in (6.1), in fact, is finite if f is a polynomial.

Applying (6.1) to our Weyl $\star$-product, we obtain

$$\begin{aligned} \xi^s \star f(\xi) &= \sum_{m=0}^{\infty} \frac{(-i\hbar)^m b_m}{m!} [\xi^{j_1}, \dots [\xi^{j_m}, \xi^s]_\hbar, \dots]_\hbar \odot d_{j_1} \dots d_{j_m} f(\xi) \\ &= \sum_{|\alpha|=0}^{\infty} \frac{(-i\hbar)^{|\alpha|} b_{|\alpha|}}{\alpha!} (\mathrm{ad}_\hbar)^\alpha (\xi^s) \odot d^\alpha f(\xi) \\ &= \sum_{m=0}^{\infty} \frac{(-i\hbar)^m b_m}{m!} \xi^s \big(\underset{\leftarrow}{\mathrm{ad}}_\hbar \odot \underset{\rightarrow}{d}\big)^m f(\xi). \end{aligned} \tag{6.2}$$

So, the following statement holds.

Theorem 6.1. *The operators of left regular representation* $(L^s f)(\xi) \stackrel{\text{def}}{=} \xi^s \star f(\xi)$ *for the Weyl* $\star$*-product are given by the formula*

$$L^s = \xi^s \mathbf{q}(-i\hbar \underset{\leftarrow}{\mathrm{ad}_\hbar} \odot \underset{\rightarrow}{d}), \tag{6.3}$$

where

$$\mathbf{q}(x) \stackrel{\text{def}}{=} \frac{x}{e^x - 1} = \sum_{m=0}^{\infty} \frac{b_m}{m!} x^m. \tag{6.4}$$

Formula (6.3) represents the star-product $\star$ via the lunar product $\odot$. On the other hand, there is another formula (3.2), which also connects these two products. Combining (6.3) and (3.2) together, we find a procedure for evaluating $\star$ and $\odot$ separately as power series in $\hbar$.

First, note that in view of (4.3) $\mathrm{ad}_\hbar^j = K^{js} \odot d_s$, and from (3.2a) we obtain

$$\mathrm{ad}_\hbar^j = \sum_{m=0}^{\infty} \frac{(-i\hbar)^m}{m+1} \sum_{|\alpha|=m} \frac{1}{\alpha!} \langle \mathrm{ad}_\hbar \rangle^\alpha K^{js} \star d^\alpha d_s. \tag{6.5}$$

Recurrently, step by step, we derive from this relation an expansion of $\mathrm{ad}_\hbar^j$ with respect to the deformation parameter $\hbar$:

$$\begin{aligned} \mathrm{ad}_\hbar^j = {} & K^{js} \star d_s - \tfrac{1}{2} i\hbar K^{\ell r} \star d_r K^{js} \star d_\ell d_s \\ & - \hbar^2 \big(\tfrac{1}{4} K^{qr} \star d_r K^{\ell s} \star d_q d_s K^{jp} \star d_\ell d_p + \tfrac{1}{6} K^{\ell q} \star d_q (K^{rp} \star d_p K^{js}) \star d_\ell d_r d_s \big) \\ & + \cdots . \end{aligned} \tag{6.6}$$

After substitution of this expansion into (3.2a) we express the lunar product $\odot$ via the star-product $\star$ as follows:

$$\begin{aligned} f\odot = {} & f \star - \tfrac{1}{2} i\hbar K^{js} \star d_s f \star d_j \\ & - \hbar^2 \big(\tfrac{1}{4} K^{\ell r} \star d_r K^{js} \star d_\ell d_s f \star d_j + \tfrac{1}{6} K^{is} \star d_s (K^{\ell r} \star d_r f) \star d_i d_\ell \big) + \cdots . \end{aligned} \tag{6.7}$$

On the other hand, formula (6.3) (or (6.2)) provides the expansion of operators L^s via the lunar product

$$\begin{aligned} L^s & = \xi^s + \frac{i\hbar}{2} \xi^s \underset{\leftarrow}{\mathrm{ad}_\hbar} \odot \underset{\rightarrow}{d} - \frac{\hbar^2}{12} \xi^s \big(\underset{\leftarrow}{\mathrm{ad}_\hbar} \odot \underset{\rightarrow}{d} \big)^2 + \ldots \\ & = \xi^s - \frac{i\hbar}{2} K^{s\ell} \odot d_\ell + \frac{\hbar^2}{12} \big(K^{rp} \odot d_p K^{s\ell} \big) \odot d_\ell d_r + O(\hbar^4) \end{aligned} \tag{6.8}$$

(the remainder contains $\hbar^4$, but no $\hbar^3$ terms, since $b_3 = 0$).

Applying (6.7) to (6.8), we obtain the following formula for L^s via the star-product:

$$\begin{aligned} L^s = {} & \xi^s - \frac{i\hbar}{2} K^{s\ell} \star d_\ell - \frac{\hbar^2}{6} K^{jr} \star d_r K^{s\ell} \star d_j d_\ell \\ & + \frac{i\hbar^3}{12} \big(K^{pm} \star d_m K^{jr} \star d_p d_r K^{s\ell} \star d_j d_\ell + \tfrac{1}{2} K^{pm} \star d_m (K^{jr} \star d_r K^{s\ell}) \star d_p d_j d_\ell \big) \\ & + O(\hbar^4). \end{aligned} \tag{6.9}$$

Note that the Weyl $\star$-product can be expressed using the set of operators of the left regular representation $L = (L^1, \dots, L^n)$ as follows:

$$f \star g = f\langle L\rangle g. \tag{6.10}$$

So, formulas (6.9), in fact, are recurrent equations for the set of operators L. By solving these equations, we obtain the final expansion of L^s in the power series in $\hbar$:

$$\begin{aligned} L^s &= \xi^s - \frac{i\hbar}{2} K^{s\ell} d_\ell + \frac{\hbar^2}{12} K^{jr} d_r K^{s\ell} d_j d_\ell \\ &- \frac{i\hbar^3}{24}\big(K^{pm} d_m K^{jr} d_p d_r K^{s\ell} d_j d_\ell + \tfrac{1}{2} K^{pm} K^{jr} d_p d_r K^{s\ell} d_m d_j d_\ell\big) + O(\hbar^4). \end{aligned} \tag{6.11}$$

The substitution of (6.11) into the right-hand side of (6.10) leads to the final expansion of the $\star$-product:

$$\begin{aligned} f \star g &= fg - \frac{i\hbar}{2} K^{sj} d_s f \cdot d_j g \\ &+ \frac{\hbar^2}{4}\big(\tfrac{1}{3} K^{\ell r} d_r K^{sj} (d_s f \cdot d_\ell d_j g - d_\ell d_s f \cdot d_j g) - \tfrac{1}{2} K^{\ell r} K^{sj} d_\ell d_s f \cdot d_r d_j g\big) \\ &+ \frac{i\hbar^3}{24}\big(K^{pj} K^{mr} d_r K^{sl} d_p d_m d_s f \cdot d_\ell d_j g - K^{pj} K^{mr} d_r K^{sl} d_\ell d_j f \cdot d_p d_m d_s g \\ &- K^{pj} d_j K^{mr} d_p d_r K^{sl} d_m d_s f \cdot d_\ell g - K^{pj} d_j K^{mr} d_p d_r K^{sl} d_s f \cdot d_m d_\ell g \\ &+ \tfrac{1}{2} K^{pj} K^{mr} K^{sl} d_p d_m d_s f \cdot d_r d_\ell d_j g + \tfrac{1}{2} K^{pj} K^{mr} d_r d_j K^{sl} d_p d_m d_s f \cdot d_\ell g \\ &- \tfrac{1}{2} K^{pj} K^{mr} d_r d_j K^{sl} d_\ell f \cdot d_p d_m d_s g - \tfrac{1}{2} K^{pj} d_j K^{mr} d_p K^{sl} d_m d_s f \cdot d_r d_\ell g \\ &- \tfrac{1}{2} K^{pj} d_j K^{mr} d_r K^{sl} d_m d_s f \cdot d_p d_\ell g + \tfrac{1}{2} K^{pj} d_j K^{mr} d_r K^{sl} d_p d_\ell f \cdot d_m d_s g \\ &- \tfrac{1}{2} K^{pj} d_\ell d_j K^{mr} K^{sl} d_m d_s f \cdot d_p d_r g\big) \\ &+ O(\hbar^4). \end{aligned} \tag{6.12}$$

This expansion up to $O(\hbar^3)$ was derived in [54]. Now we have explicit formulas for the $\star$-product in all orders.

After this, the expansion for the lunar product can readily be obtained from (6.7) and (6.12)

$$f \odot g = fg - \frac{\hbar^2}{12}\big(K^{\ell r} d_r K^{sj} d_\ell d_s f \cdot d_j g + \tfrac{1}{2} K^{\ell r} K^{sj} d_\ell d_s f \cdot d_r d_j g\big) + O(\hbar^4). \tag{6.13}$$

(all odd powers of $\hbar$ are absent).

The substitution of this expansion into the quantum Jacobi conditions (5.2) leads to closed (highly nonlinear) equations for the quantum tensor K. The first terms in the $\hbar$-expansion for these equations are the following:

$$\begin{aligned} &\mathop{\mathfrak{S}}_{(j_1,j_2,j_3)} K^{j_3\ell} d_\ell K^{j_2 j_1} - \frac{\hbar^2}{12} \mathop{\mathfrak{S}}_{(j_1,j_2,j_3)} \big(K^{pr} d_r K^{sq} d_p d_s K^{j_3\ell} d_q d_\ell K^{j_2 j_1} \\ &\qquad + \tfrac{1}{2} K^{pr} K^{sq} d_p d_s K^{j_3\ell} d_r d_q d_\ell K^{j_2 j_1}\big) + O(\hbar^4) = 0 \end{aligned} \tag{6.14}$$

(all odd powers of $\hbar$ are absent).

Thus, we have proved the following statement.

Theorem 6.2. (i) *Formulas* (3.2) *and* (6.3) *determine an explicit expansion of the Weyl star product* $\star$ *in the power series in* $\hbar$ *via the quantum* $\star$*-tensor* K. *First terms of this expansion are given by* (6.12).

(ii) *Formulas* (3.2) *and* (6.3) *determine the expansion of the lunar product* $\odot$ *as well. The first terms are given in* (6.13).

(iii) *Any skew-symmetric quantum tensor* $K \in V_\hbar^2(\mathcal{M})$ *satisfying equations* (6.14) *determines via* (6.12) *a unique associative* $\star$*-product, which makes* $\mathcal{M}$ *a quantum Weyl-manifold with* $\star$*-tensor* K.

Remark 6.1. If we seek the solution of (6.14) in the form (2.5), then for the quantum corrections $P_{(1)}, P_{(2)}, \ldots$ to the Poisson tensor P we obtain a chain of linear equations of the following type:

$$[\![P, P_{(m)}]\!] = \Gamma_{(m)}, \qquad m = 1, 2, \ldots . \tag{6.15}$$

Here $[\![\cdot, \cdot]\!]$ are the classical Schouten brackets, and the right-hand sides $\Gamma_{(m)}$ for each fixed m are given explicitly via $P, P_{(1)}, \ldots, P_{(m-1)}$. The formula for $\Gamma_{(1)}$:

$$\Gamma_{(1)} = \frac{1}{12} \underset{(j_1, j_2, j_3)}{\mathfrak{S}} \left(P^{pr} d_r P^{sq} d_p d_s P^{j_3\ell} d_q d_\ell P^{j_2 j_1} + \tfrac{1}{2} P^{pr} P^{sq} d_p d_s P^{j_3\ell} d_r d_q d_\ell P^{j_2 j_1}\right)$$

was obtained in [54]. Now we have represented the explicit procedure for calculation of all $\Gamma_{(m)}$, $m \geqslant 1$.

Note that formula (6.12) after the substitution of expansion (2.5) reads

$$\begin{aligned}
f \star g = {} & fg - \frac{i\hbar}{2} P^{sj} d_s f \cdot d_j g \\
& + \hbar^2 \left(\tfrac{1}{12} P^{\ell r} d_r P^{sj} (d_s f \cdot d_\ell d_j g - d_\ell d_s f \cdot d_j g) - \tfrac{1}{8} P^{\ell r} P^{sj} d_\ell d_s f \cdot d_r d_j g\right) \\
& + i\hbar^3 \big(\tfrac{1}{24} P^{pj} P^{mr} d_r P^{sl} d_p d_m d_s f \cdot d_\ell d_j g - \tfrac{1}{24} P^{pj} P^{mr} d_r P^{sl} d_\ell d_j f \cdot d_p d_m d_s g \\
& \quad - \tfrac{1}{24} P^{pj} d_j P^{mr} d_p d_r P^{sl} d_m d_s f \cdot d_\ell g - \tfrac{1}{24} P^{pj} d_j P^{mr} d_p d_r P^{sl} d_s f \cdot d_m d_\ell g \\
& \quad + \tfrac{1}{48} P^{pj} P^{mr} P^{sl} d_p d_m d_s f \cdot d_r d_\ell d_j g + \tfrac{1}{48} P^{pj} P^{mr} d_r d_j P^{sl} d_p d_m d_s f \cdot d_\ell g \\
& \quad - \tfrac{1}{48} P^{pj} P^{mr} d_r d_j P^{sl} d_\ell f \cdot d_p d_m d_s g - \tfrac{1}{48} P^{pj} d_j P^{mr} d_p P^{sl} d_m d_s f \cdot d_r d_\ell g \\
& \quad - \tfrac{1}{48} P^{pj} d_j P^{mr} d_r P^{sl} d_m d_s f \cdot d_p d_\ell g + \tfrac{1}{48} P^{pj} d_j P^{mr} d_r P^{sl} d_p d_\ell f \cdot d_m d_s g \\
& \quad - \tfrac{1}{48} P^{pj} d_\ell d_j P^{mr} P^{sl} d_m d_s f \cdot d_p d_r g - \tfrac{1}{2} P^{sj}_{(1)} d_s f \cdot d_j g\big) \\
& + O(\hbar^4).
\end{aligned} \tag{6.16}$$

This formula expresses the Weyl $\star$-product in terms of the Poisson tensor P, and its quantum corrections $P_{(m)}$.

The solvability of system (6.15) for quantum corrections, in general, is an open question till now. Of course, for a wide class of particular cases, the positive answer is known. For the case of quadratic Poisson tensor P, the system (6.15) (and the initial equation (6.14)) is equivalent to the quantum Yang–Baxter equation (see [21, 22, 2]). Perhaps, the results [97] could clarify this problem for the general case. Note that the remarkable and unexpected deformation formula discovered in [60] in three leading orders coincides with (6.16), but in the order $\hbar^3$ this deformation formula seems to be of a non-Weyl type and does not generate a solution $P_{(1)}$ of equation (6.15) for $m = 1$.

We stress the following: if equations (6.14) or (6.15) are solvable and so, the quantum $\star$-tensor K does exist, than the corresponding $\star$-product algebra has representations by $\hbar$-pseudodifferential operators (see in [54], as well in Sect. 9 below). In this case, in view of the general equivalence theorem [60], all other possible formal star-products on $\mathcal{M}$ would also have operator representations. So, one can say that in this case *the classical mechanics on* $\mathcal{M}$ *is deformed* to a quantum mechanics, or more precisely, *to a semiclassical mechanics.* And back: if some star-product over $\mathcal{M}$ has a representation by $\hbar$-pseudodifferential operators, then equations (6.14) and (6.15) are solvable.

7. Properties of quantum vector fields

From (6.13) one can derive the expansion of quantum vector fields (3.3).

Theorem 7.1. *The quantum vector fields have the expansion in even powers of* $\hbar$*:*

$$a^{\frown} = a + \hbar^2 a_{(1)} + \hbar^4 a_{(2)} + \dots,$$

where $a = a^q d_q$*, and all coefficients* $a_{(k)}$ *are explicitly known differential operators of order* $2k+1$*. In particular,*

$$a_{(1)} = \tfrac{1}{24} P^{\ell r} P^{sj} d_r d_s a^q \cdot d_q d_\ell d_j + \tfrac{1}{12} P^{\ell r} \cdot d_r P^{sj} \cdot d_\ell d_j a^q \cdot d_q d_s.$$

Now let us note that in view of Theorem 6.2 the Weyl $\star$-product is generated by the quantum tensor K only. So, any variation of the star-product is determined by a variation of K. Let $\underset{\varepsilon}{\star}$ be a family of Weyl star-products, starting from $\star$ at $\varepsilon = 0$. Then the first variation of K can be defined as

$$\delta K^{j\ell} = \frac{d}{d\varepsilon}\Big(\frac{i}{\hbar}(\xi^j \underset{\varepsilon}{\star} \xi^\ell - \xi^\ell \underset{\varepsilon}{\star} \xi^j)\Big)\Big|_{\varepsilon=0}.$$

The first variation of the $\star$-product is given by

$$\frac{d}{d\varepsilon}(f \underset{\varepsilon}{\star} g)\Big|_{\varepsilon=0} = \hbar \cdot \Delta_{\delta K}(f, g), \tag{7.1}$$

where the bidifferential operator $\Delta_{\delta K}$ is determined only by K and δK.

Lemma 7.1. *The variation* (7.1) *of the* $\star$*-product generated by a variation* δK *of the* $\star$*-tensor* K *can be represented as the sum*

$$\Delta_{\delta K}(f, g) = -\frac{i\hbar}{2}(\delta K^{s\ell} \odot d_s f) \odot d_\ell g + \hbar^2 u_{\delta K}(f, g).$$

Here the first summand is skew symmetric in f, g *and pure imaginary, the second summand is symmetric in* f, g *and real. The expansion of* $u_{\delta K}(f, g)$ *in* $\hbar$ *contains only even powers and is explicitly known from* (6.12)*; the leading term is the following*:

$$u_{\delta K}(f, g) = \tfrac{1}{12}\big(K^{\ell r} d_r (\delta K)^{sj} + (\delta K)^{\ell r} d_r K^{sj}\big)\big(d_s f \cdot d_\ell d_j g - d_\ell d_s f \cdot d_j g\big)$$
$$- \tfrac{1}{4} K^{\ell r} (\delta K)^{sj} d_\ell d_s f \cdot d_r d_j g + O(\hbar^2).$$

We have the following two basic theorems that describe properties of quantum vector fields.

Theorem 7.2. *Any quantum vector field a satisfies the following analog of the Leibniz rule:*

$$a^\frown(f \star g) = a^\frown(f) \star g + f \star a^\frown(g) - \hbar \cdot \Delta_{[\![K,a]\!]_\hbar}(f, g). \tag{7.2}$$

So, in general, quantum vector fields are not derivations of the $\star$-algebra (but any quantum Euler–Poisson vector field $a = \mathrm{ad}_\hbar(f) = -[\![K, f]\!]_\hbar$ is such a derivation).

Proof. By definition (3.1), we have

$$a^\frown(f \star g)(\xi) = \lim \frac{1}{\varepsilon}\big((f \star g)\langle \xi + \varepsilon a(\xi)\rangle - (f \star g)(\xi)\big). \tag{7.3}$$

Let us consider the product $\underset{\varepsilon}{\star}$ generated by the noncommutative variables $\xi^s + \varepsilon a^s(\xi)$, $s = 1, \dots, n$. Note that

$$\begin{aligned}
\big[\xi^\ell + \varepsilon a^\ell(\xi), \xi^s + \varepsilon a^s(\xi)\big]_\hbar &= K^{\ell s}(\xi) + \varepsilon\big([\xi^\ell, a^s]_\hbar - [\xi^s, a^\ell]_\hbar\big) + O(\varepsilon^2) \\
&= K^{\ell s}(\xi + \varepsilon a) - \varepsilon a^\frown(K^{\ell s}) \\
&\quad + \varepsilon(K^{\ell q} \odot d_q a^s - K^{sq} \odot d_q a^\ell) + O(\varepsilon^2) \\
&= K_\varepsilon^{\ell s}\langle \xi + \varepsilon a(\xi)\rangle + O(\varepsilon^2).
\end{aligned}$$

Here the deformed $\underset{\varepsilon}{\star}$-tensor K_ε is given by

$$K_\varepsilon = K + \varepsilon[\![K, a]\!]_\hbar + O(\varepsilon^2),$$

where $[\![K, a]\!]_\hbar$ is the quantum bracket between the quantum $\star$-tensor K and the quantum vector fields a. So, by definition (7.1)

$$f \underset{\varepsilon}{\star} g = f \star g + \varepsilon\hbar\Delta_{[\![K,a]\!]_\hbar}(f, g) + O(\varepsilon^2).$$

After the substitution to (7.3) we obtain

$$\begin{aligned}
a^\frown(f \star g) &= \lim \Big[\frac{1}{\varepsilon}(f \underset{\varepsilon}{\star} g)\langle \xi + \varepsilon a\rangle - (f \star g)(\xi) - \varepsilon\hbar\Delta_{[\![K,a]\!]_\hbar}(f, g)\Big] \\
&= \lim \frac{1}{\varepsilon}\big[f\langle \xi + \varepsilon a\rangle \star g\langle \xi + \varepsilon a\rangle - (f \star g)(\xi)\big] - \hbar\Delta_{[\![K,a]\!]_\hbar}(f, g) \\
&= \lim \frac{1}{\varepsilon}\big[\big(f + \varepsilon a^\frown(f)\big) \star \big(g + \varepsilon a^\frown(g)\big) - (f \star g)\big] - \hbar\Delta_{[\![K,a]\!]_\hbar}(f, g) \\
&= a^\frown(f) \star g + f \star a^\frown(g) - \hbar\Delta_{[\![K,a]\!]_\hbar}(f, g).
\end{aligned}$$

The proof is complete. $\square$

Theorem 7.3. *Under the change of local coordinates $\xi = \varphi(\xi')$, the components of a quantum vector field change as follows:*

$$a^s\langle\varphi\rangle = a'^\ell \odot d'_\ell \varphi^s. \tag{7.4}$$

Here $a = \{a^s\}$ are the components of the quantum vector field in local coordinates ξ, $\{a'^\ell\}$ are the components of the same field in local coordinates ξ', $d'_\ell = d/d\xi'^\ell$, and the lunar product $\odot$ in (7.4) is taken in the coordinates ξ'.

Proof. Let us denote by V_ε^φ the transformation

$$V_\varepsilon^\varphi\colon f \to f\langle\varphi\rangle_\varepsilon,$$

where the symmetrization $\langle\cdot\rangle_\varepsilon$ is taken with respect to the product $\underset{\varepsilon}{\star}$ generated by the non-commutative variables $\xi' + \varepsilon a(\xi')$ (see in the proof of Theorem 7.2 above). Then, by definition, there should be the equality

$$V_0^\varphi\big(f\langle\xi + \varepsilon a\rangle\big) = V_\varepsilon^\varphi(f)\langle\xi' + \varepsilon a'\rangle + O(\varepsilon^2). \tag{7.5}$$

On the right of (7.5), we have

$$V_\varepsilon^\varphi(f)\langle\xi' + \varepsilon a'\rangle = f\langle\varphi(\xi' + \varepsilon a')\rangle = f\langle\varphi(\xi') + \varepsilon a'^\frown(\varphi) + O(\varepsilon^2)\rangle.$$

On the left of (7.5) we have

$$V_0^\varphi\big(f\langle\xi + \varepsilon a\rangle\big) = f\langle\varphi(\xi') + \varepsilon a\langle\varphi\rangle\rangle.$$

The comparison of these two relations leads to (7.4). The theorem is proved. $\square$

Corollary 7.1. *Under the change of local coordinates $\xi = \varphi(\xi')$ the components of the $\star$-tensor K (2.4) change as follows:*

$$K^{s\ell}\langle\varphi\rangle = (K'^{qp} \odot d'_q\varphi^s) \odot d'_p\varphi^\ell$$

(with the same notation as in (7.4)).

8. Quantum version of the de Rham complex

Let us try to define the space of quantum differential forms according to the usual idea of duality (or pairing) between forms and vector fields. Now we have to consider, of course, quantum vector fields. Our version of quantum pairing is based on the multi-component lunar product (3.2b).

Let $\omega = \{\omega_{j_1\dots j_m}\}$ be a set of functions skew-symmetric in the indices $j_1, \dots, j_m$. For any quantum vector fields $a_1, \dots, a_m$, we define

$$\omega(a_1, \dots, a_m) \overset{\text{def}}{=} (a_1^{j_1} \vee \dots \vee a_m^{j_m}) \odot \omega_{j_1\dots j_m}.$$

Thus we obtain a set of *quantum forms* of all degrees $m = 1, 2, \dots$.

A differential in the complex of quantum forms can be defined by analogy with the classical Cartan formula:

$$\begin{aligned} d\omega(a_1, \dots, a_{m+1}) &= \sum_{k=1}^{m+1} (-1)^{k+1} a_k{}^\frown\big(\omega(a_1, \dots, \breve{a}_k, \dots, a_m)\big) \\ &\quad - \sum_{k,s=1}^{m+1} (-1)^{k+s+1} \omega\big([a_k, a_s]_\hbar, a_1, \dots, \breve{a}_k, \dots, \breve{a}_s, \dots, a_m\big). \end{aligned} \tag{8.1}$$

Theorem 8.1. *The differential* (8.1) *of the quantum form* ω *is given by*

$$(d\omega)_{j_1\dots j_{m+1}} = \sum_{k=1}^{m+1}(-1)^{k+1}d_{j_k}\omega_{j_1\dots\check{j}_k\dots j_{m+1}}.$$

The proof of this theorem uses the following generalization of formula (3.4).

Lemma 8.1.

$$a^\frown\big((g_1\vee\cdots\vee g_m)\odot f\big) = \sum_{r=1}^{m}\big(g_1\vee\cdots\vee a^\frown(g_r)\vee\cdots\vee g_m\big)\odot f + (g_1\vee\cdots\vee g_m\vee a^s)\odot d_s f.$$

By Theorem 8.1, in the Weyl local coordinates the differential of a quantum form looks like the differential of the classical form.

In particular, exact quantum forms of degree 1 are just usual differentials of functions acting on quantum vector fields by the formula

$$df(a) = a^\frown(f).$$

The first quantum cohomology with respect to differential (8.1) coincides with the classical de Rham cohomology.

9. Quantum phase space over a Poisson manifold

Let $\mathcal{M}$ be a Poisson manifold with Poisson tensor P, and let K be a solution of equation (6.14) admitting an expansion of type (2.5). Then K generates the Weyl $\star$-product (6.12) over $\mathcal{M}$.

This $\star$-product is well-defined on the space of functions regularly depending on $\hbar\to 0$ (as a formal power series). Actually, *such a regular picture is not sufficient for quantization*, since solutions of any, even simplest, quantum equations are not regular and automatically oscillate like $\exp\{iS/\hbar\}$ or in a more complicated manner as $\hbar\to 0$. For instance, the problem of quantization of the pseudogroup of Poisson transformations, even in the symplectic case, is not solvable in any algebra of regular, i.e., not $\hbar$-oscillating functions. Therefore such oscillations (on which all powers of operators $\hbar d_j$ are bounded as $\hbar\to 0$) are absolutely inevitable and should be included into a space $\mathcal{F}_\hbar(\mathcal{M})$ of observables over $\mathcal{M}$, see [51, 53, 54]. The $\star$-product on the extended algebra $\mathcal{F}_\hbar(\mathcal{M})$ is represented by an $\hbar$-expansion

$$(f\star g)(\xi) = \sum_m \hbar^m f T_m^\hbar(i\hbar\overleftarrow{d}, \xi, i\hbar\overrightarrow{d})g, \tag{9.0}$$

where $d = \partial/\partial\xi$, and $T_m^\hbar$ are symbols of some integral $\hbar$-Fourier operators over $\mathcal{M}$.

The Weyl star-products of type (9.0) over symplectic manifolds were constructed in [53]. The general Poisson case is a more difficult problem.

Geometrically, the appearance of $\hbar$-oscillations in observables f,g means that in addition to the Poisson manifold $\mathcal{M}$, one should consider a phase space $\mathcal{E}$ of doubled dimension in which the manifold $\mathcal{M}\subset\mathcal{E}$ plays the role of a "configuration subspace," and the oscillations are controled by a dual "momenta subspace." Following this idea, a phase space $\mathcal{E}$ was defined in [51] as a symplectic manifold with a pair of polar Poisson mappings:

$$\begin{aligned} &\ell\colon \mathcal{E} \to \mathcal{M}, \qquad r\colon \mathcal{E} \to \mathcal{M}^{(-)}, && \{\ell, r\}_{\mathcal{E}} = 0, \\ &\mathcal{M} \text{ is a Lagrangian submanifold in } \mathcal{E}, && \ell\big|_{\mathcal{M}} = r\big|_{\mathcal{M}} = \mathrm{id}\,. \end{aligned} \tag{9.1}$$

Here the minus in $\mathcal{M}^{(-)}$ means that the Poisson tensor on $\mathcal{M}$ just changes the sign; the brackets $\{\ell, r\}_{\mathcal{E}}$ are the Poisson brackets between the components of ℓ and r on $\mathcal{E}$.

In the case $\mathcal{M} = \mathbf{g}^*$, where $\mathbf{g}$ is a Lie algebra and the Poisson tensor on $\mathbf{g}^*$ is linear, the phase space $\mathcal{E}$ is isomorphic to the cotangent bundle $T^*G \approx \mathbf{g}^* \times G$, where G is the Lie group corresponding to $\mathbf{g}$. In this case, the mappings ℓ and r are given by the Lie construction of right and left group translations of covectors.

In fact, Sofus Lie considered [65] the nonlinear Poisson brackets as well (and even earlier than the linear ones). He found the solution of (9.1) in neighborhoods of points on $\mathcal{M}$, where the rank of the Poisson tensor is maximal (just applying a generalization of the Darboux theorem; see explanations in [26]).

In the general degenerate Poisson case, *explicit formulas* for ℓ, r were discovered in [42] (see also [52, formula (6)]) and in [90] in two versions: of first and second kind coordinates, which correspond to the Weyl and to the normal ordering of noncommuting variables. Then this basic observation was developed in [43, 44] and in [91] up to the notion of symplectic groupoid over $\mathcal{M}$. Namely, the phase manifold $\mathcal{E}$ can be endowed with a groupoid structure whose *reduction mappings* ("target" and "source") are ℓ and r; see details also in [54, 66, 98] and in Weinstein's paper in this volume.

On the quantum level, $\mathcal{E}$ and $\mathcal{M}$ are supposed to be quantum manifolds, and the mappings ℓ and r in (9.1) should be replaced by commuting *reduction homomorphisms* of quantum function algebras

$$\begin{aligned} &\hat{\ell}^*\colon \mathcal{F}_\hbar(\mathcal{M}) \to \mathcal{F}_\hbar(\mathcal{E}), && \hat{r}^*\colon \mathcal{F}_\hbar(\mathcal{M})^{(-)} \to \mathcal{F}_\hbar(\mathcal{E}), \\ &\hat{j}_{\mathcal{M}} \cdot \hat{\ell}^* = \hat{j}_{\mathcal{M}} \cdot \hat{r}^* = I, && [\hat{\ell}^*(f), \hat{r}^*(g)] = 0, \quad \forall\, f, g. \end{aligned} \tag{9.2}$$

The sign minus means the inverse ordering of the multipliers in the star-product, and $\hat{j}_{\mathcal{M}}$ denotes a quantum restriction of functions from $\mathcal{E}$ onto $\mathcal{M}$.

Also note that functions on the phase space $\mathcal{E}$ are considered as symbols of $\hbar$-pseudo-differential operators acting over $\mathcal{M}$. That is why, in addition to (9.2), the algebra $\mathcal{F}_\hbar(\mathcal{E})$ is assumed to act on $\mathcal{F}_\hbar(\mathcal{M})$ (as on a vector space) and the reduction homomorphisms (9.2) intertwine this action with the left and right regular representations:

$$(\hat{\ell}^* f) \circ g = f \star g, \qquad (\hat{r}^* f) \circ g = g \star f. \tag{9.2a}$$

Explicit formulas for quantum reduction mappings $\hat{\ell}^*$ and $\hat{r}^*$ were obtained in [43], using, as a basic object, the Weyl $\star$-product on the space of regular (nonoscillating) functions over $\mathcal{M}$.

Now, Theorem 6.2 shows how to construct such a "basic" $\star$-product over Poisson manifold $\mathcal{M}$ starting from a solution K of equation (6.14). Thus, we obtain the following theorem.

Theorem 9.1. *The construction of the formal Weyl $\star$-product via Theorem* 6.2 *in combination with the result of* [43] (*for details, see* [54, p. 249–263 of the English edition]) *generates the* *explicit construction of the Weyl $\star$-product of type* (9.0) *in the extended algebra* $\mathcal{F}_\hbar(\mathcal{M})$, *as well, the construction of quantum reduction homomorphisms with properties* (9.2), (9.2a).

Note that in (9.2) the relations on the second line involve just the classical restriction mapping $\hat{j}_{\mathcal{M}} = j_{\mathcal{M}}$, $j_{\mathcal{M}}(f) \stackrel{\text{def}}{=} f\big|_{\mathcal{M}}$, if on the space $\mathcal{E}$ the star-product is defined by the normal ordering quantization (with respect to the real "vertical" polarization given by the "anchor" projection $\mathcal{E} \to \mathcal{M}$). If over $\mathcal{E}$ one considers some other star-product, say, the Weyl product, then the classical restriction mapping $j_{\mathcal{M}}$ must be replaced by the quantum one $\hat{j}_{\mathcal{M}}$, which is the composition of $j_{\mathcal{M}}$ and the morphism transporting Weyl-symbols to the normal ordering symbols.

Note also, that we can change the $\star$-product on $\mathcal{M}$ as well; i.e., in (9.2) we can consider some other star-product in $\mathcal{F}_\hbar(\mathcal{M})$ instead of the Weyl product. An interesting case is again the star-product corresponding to the normal ordering in some local coordinates on $\mathcal{M}$. These new coordinates are functions of old Weyl coordinates, and so, this new "normal" star-product can be evaluated explicitly just by using general transformation formulas between the Weyl and the "normal" operator calculus; all coefficients of $\hbar$-expansions are known in an exact closed form (see [41] or [54, Appendix 1]). So, in (9.2) we can easily replace the Weyl star product over $\mathcal{M}$ by any other product which is convenient for calculations.

Now we consider one important special situation in which the normal ordering over $\mathcal{M}$ is preferable in comparison with the Weyl symmetrization. More precisely, it will be the normal ordering corresponding to "creation and annihilation operators," or to a quantum Kählerian polarization over $\mathcal{M}$.

10. Quantum complexification and vacuum submanifold

The *partial complex structure* on $\mathcal{M}$ [16, 69, 31] is defined by an atlas with local coordinates (A, C), where $A = (A^1, \dots, A^k)$ are real, $C = (C_1, \dots, C_d)$ are complex; and changes of complex coordinates on intersections of charts are assumed to be holomorphic. (Bellow all complex coordinates are supposed to belong to a neighborhood of the origin in $\mathbb{C}$.)

So, now instead of real Weyl coordinates $\xi = (\xi^1, \dots, \xi^n)$ on $\mathcal{M}$ we shall use the partial complex coordinates $(A^1, \dots, A^k, C_1, \dots, C_d)$, where $k + 2d = n$.

The *complexification* $\mathcal{M}^{\mathbb{C}}$ of $\mathcal{M}$ is a tubular neighborhood of $\mathcal{M}$ with local coordinates (B, A, C), where $B = (B_1, \dots, B_d)$, the coordinates C, $\overline{B}$ are holomorphic, and A are real; the equation $C = \overline{B}$ determines the embedding of $\mathcal{M}$ into $\mathcal{M}^{\mathbb{C}}$; there is an involution $\mathcal{M}^{\mathbb{C}} \to \mathcal{M}^{\mathbb{C}}$, $(B, A, C) \mapsto (\overline{C}, A, \overline{B})$, whose fixed points constitute the real manifold $\mathcal{M} \subset \mathcal{M}^{\mathbb{C}}$.

Locally, each function $f = f(\overline{C}, A, C)$ on $\mathcal{M}$ can be holomorphically extended to a functions $f(B, A, C)$ on $\mathcal{M}^{\mathbb{C}}$, and the conjugate functions $\overline{f}$ is extended to $\overline{f}(C, A, B)$, and thus the complex conjugation is consistent with the involution on $\mathcal{M}^{\mathbb{C}}$ (see [35, 84]).

A partial complex structure is *consistent* with Poisson brackets $\{\cdot\,,\,\cdot\}_{\mathcal{M}}$ on $\mathcal{M}$, if purely holomorphic functions $g = g(C)$ form a commutative Poisson subalgebra. A consistent complex structure is called *complex polarization*, if partially holomorphic functions (annuled by $\partial/\partial B$) form a Poisson subalgebra. This structure is called *partial Kählerian* if the matrix of Poisson brackets $(1/i)\{B_j, \overline{B}_\ell\}_{\mathcal{M}}$ is positive almost everywhere.

We say that a complex polarization over a Poisson manifold $\mathcal{M}$ possesses a *vacuum*, if there is a submanifold $\mathcal{M}_0 \subset \mathcal{M}$ locally given by equations $\mathcal{M}_0 = \{C = 0\}$ and such that Poisson brackets of any two partially holomorphic functions vanish on $\mathcal{M}_0$. The dimension of the vacuum submanifold is $\dim \mathcal{M}_0 = k$.

About other possible ways to introduce the notion of polarization over a Poisson manifold see [38, 39, 86, 92].

The *quantum Kählerian polarization* over a Poisson manifold $\mathcal{M}$ is a partial Kählerian polarization with a vacuum together with a star-product $*$ possessing the following properties:

(a) $\quad \overline{f * g} = \bar{g} * \bar{f}$;

(b) $\quad (i/\hbar)(f * g - g * f) = \{f, g\}_{\mathcal{M}} + O(\hbar)$

for any two functions f, g which are regular as $\hbar \to 0$;

(c) $\quad f * g = fg$

for any f and any purely holomorphic function g;

(d) partially holomorphic functions form a closed subalgebra $\mathcal{F}_\hbar^{(1)}(\mathcal{M}) \subset \mathcal{F}_\hbar(\mathcal{M})$ with respect to the star-product $*$;

(e) $\quad (f * g)\big|_{\mathcal{M}_0} = f\big|_{\mathcal{M}_0} \cdot g\big|_{\mathcal{M}_0}$

for any $g \in \mathcal{F}_\hbar^{(1)}(\mathcal{M})$, where $\mathcal{M}_0$ is a vacuum submanifold in $\mathcal{M}$.

In this case the star-product $*$ will be called the *normal product* over $\mathcal{M}$ (with respect to the given quantum polarization).

Remark 10.1. Usually, the coordinates C_j correspond to annihilation operators $\mathbf{C}_j$, the coordinates B_j correspond to creation operators $\mathbf{B}_j = \mathbf{C}_j^*$, and condition (c) means the normal ordering of these operators: the annihilations $\mathbf{C}$ are the first, the creations $\mathbf{B}$ are the last. If operators $\mathbf{A}^\mu = (\mathbf{A}^\mu)^*$ correspond to the real coordinates A, then we can say that the function $f(B, A, C)$ is the *normal symbol* of the operator

$$\hat{f} = f(\overset{3}{\mathbf{B}}, \overset{2}{\mathbf{A}}, \overset{1}{\mathbf{C}}).$$

So, in this notation $\hat{f}'\widehat{f''} = \widehat{f' * f''}$.

From the previous discussion in Section 9 we obtain the following result.

Theorem 10.1. *Suppose that a Poisson manifold $\mathcal{M}$ is endowed with a partial Kählerian structure with a vacuum, and also there is a star-product over $\mathcal{M}$, say, the Weyl $\star$-product via Theorem 6.2. Suppose that the holomorphic coordinates C_j mutually commute with respect to $\star$, as well A^μ, and the coordinates A, C generate a closed subalgebra with respect to $\star$. Then over $\mathcal{M}$ there is an explicit construction of the normal product $*$ on the space of functions which are formal power series in $\hbar$; this is the unique star-product over $\mathcal{M}$ such that the correspondence*

$$\text{“normal symbol”} \mapsto \text{“Weyl symbol,”} \qquad A^\alpha B^\beta C^\gamma \mapsto B^\beta \star A^\alpha \star C^\gamma$$

is a homomorphism of $*$-algebra to $\star$-algebra.

This product $*$ can be also extended to a space $\mathcal{F}_\hbar(\mathcal{M})$ of functions with singular exponential dependence in $\hbar$ like $\exp\{S/\hbar\}$. *Namely, we have*

$$f * g = f(\overset{3}{L_B}, \overset{2}{L_A}, \overset{1}{L_C})g = g(\overset{1}{R_B}, \overset{2}{R_A}, \overset{3}{R_C})f, \tag{10.1}$$

where L_B, L_A, L_C (or R_B, R_A, R_C) are operators of the left (or right) regular representation for the normal $*$-product; all these operators are of the following type:

$$S = \mathcal{S}\big(\overset{2}{B}, \hbar\, \overset{1}{\partial/\partial B};\ \overset{2}{A}, \hbar\, \overset{1}{\partial/\partial A};\ \overset{2}{C}, \hbar\, \overset{1}{\partial/\partial C}\big), \tag{10.2}$$

with symbols

$$\mathcal{S} = \mathcal{S}^{(0)} + \hbar\mathcal{S}^{(1)} + \hbar^2\mathcal{S}^{(2)} + \cdots \quad \textit{expansion in power series in } \hbar.$$

The symbols $\mathcal{L}_B$, $\mathcal{L}_A$, $\mathcal{L}_C$ of operators L_B, L_A, L_C of the left regular representation, as well symbols $\mathcal{R}_B$, $\mathcal{R}_A$, $\mathcal{R}_C$ of operators R_B, R_A, R_C of the right regular representation are evaluated explicitly.

Of course, the symbols $\mathcal{L}_B = B$ and $\mathcal{R}_C = C$ are obviously given. Formulas for other symbols $\mathcal{L}_A$, $\mathcal{L}_C$ and $\mathcal{R}_B$, $\mathcal{R}_A$ are too cumbersome to reproduce them here. We refer to [55] for details and only mention that these formulas are obtained almost in the same way as in the Weyl case. Below in Section 15 we precisely describe the triple of *leading* symbols $\ell^{\mathbb{C}} = (\mathcal{L}_B^{(0)}, \mathcal{L}_A^{(0)}, \mathcal{L}_C^{(0)})$ and $r^{\mathbb{C}} = (\mathcal{R}_B^{(0)}, \mathcal{R}_A^{(0)}, \mathcal{R}_C^{(0)})$.

Note also that Theorem 10.1 implicitly contains a statement about the existence of a classical phase space $\mathcal{E}^{\mathbb{C}}$ over the complexification $\mathcal{M}^{\mathbb{C}}$ of the Poisson manifold $\mathcal{M}$. This phase space is just a partial complexification of the phase space $\mathcal{E}$ over $\mathcal{M}$.

Corollary 10.1. *Leading symbols of operators of left and right regular representations for the $*$-algebra generate polar Poisson mappings:*

$$\begin{aligned} &\ell^{\mathbb{C}}\colon \mathcal{E}^{\mathbb{C}} \to \mathcal{M}^{\mathbb{C}}, \qquad r^{\mathbb{C}}\colon \mathcal{E}^{\mathbb{C}} \to \mathcal{M}^{\mathbb{C}(-)}, \qquad \{\ell^{\mathbb{C}}, r^{\mathbb{C}}\} = 0, \\ &\mathcal{M}^{\mathbb{C}} \textit{ is a Lagrangian submanifold in } \mathcal{E}^{\mathbb{C}}, \qquad \ell^{\mathbb{C}}\big|_{\mathcal{M}^{\mathbb{C}}} = r^{\mathbb{C}}\big|_{\mathcal{M}^{\mathbb{C}}} = \mathrm{id}. \end{aligned} \tag{10.3}$$

In fact, the manifold $\mathcal{E}^{\mathbb{C}}$ can be endowed with a groupoid structure (with reduction mappings $\ell^{\mathbb{C}}, r^{\mathbb{C}}$) in the same way as in [44]. As we see in the next section, this partial complex groupoid $\mathcal{E}^{\mathbb{C}}$ is closely related to geometric objects which control irreducible representations of the quantum manifolds $\mathcal{M}$.

11. Explicit construction of irreducible representations of the star-product algebra

Let $\mathcal{M}$ be endowed with a quantum Kählerian polarization. For each $a \in \mathcal{M}_0$ let us denote by j_a the restriction mappings $j_a(f) \overset{\text{def}}{=} f(a)$, or in other notation

$$j_a\colon f(B, A, C) \mapsto f(0, a, 0). \tag{11.1}$$

For arbitrary multi-indices $\alpha, \beta \in \mathbb{Z}_+^d$, we define the numbers

$$k(a)_{\alpha\beta} \overset{\text{def}}{=} j_a(\overline{B}^{\alpha} * B^{\beta}). \tag{11.2}$$

These numbers are explicitly known if one knows only permutation relations between C, B, A, i.e., if one knows the quantum $*$-tensor (for detail see [55]).

For any $a \in \mathcal{M}_0$, let us introduce the "reproducing kernel":

$$\mathcal{K}_a(\bar{z}, z) \stackrel{\text{def}}{=} j_a(\bar{e}_z * e_z) = \sum_{|\alpha|,|\beta|\geqslant 0} \frac{\bar{z}^\alpha z^\beta}{\hbar^{|\alpha|+|\beta|}\alpha!\,\beta!} k(a)_{\alpha\beta}, \tag{11.3}$$

where

$$e_z(B) \stackrel{\text{def}}{=} \exp(zB/\hbar).$$

We say that the normal product over $\mathcal{M}$ *admits a vacuum* if for each $a \in \mathcal{M}_0$ the following condition holds:

(f) the infinite Hermitian matrix $k(a)$ (11.2) is positive definite, and the series (11.3) has a nonzero radius of convergence.

Using the inverse matrix $k(a)^{-1}$, we can define the Hilbert space $\mathcal{P}_a$ of antiholomorphic distributions with the following inner product:

$$\begin{aligned} g(\bar{z}) &= \sum_{|\alpha|\geqslant 0} \frac{\bar{z}^\alpha}{\alpha!} g_\alpha \in \mathcal{P}_a, \\ (g, g')_{\mathcal{P}_a} &\equiv \sum_{|\alpha|,|\beta|\geqslant 0} \hbar^{|\alpha|+|\beta|}\big(k(a)^{-1}\big)^{\alpha\beta}\, g'_\alpha\, \bar{g}_\beta. \end{aligned} \tag{11.3a}$$

Note that the reproducing kernel $\mathcal{K}_a$ is just the integral kernel of the unity operator I in the Hilbert space $\mathcal{P}_a$.

Remark 11.1. In the notation of Remark 10.1 the restriction mapping (11.1) is the mean value of the operator on a vacuum vector $\mathfrak{P}_a(0)$, i.e.,

$$j_a(f) = \big(\mathfrak{P}_a(0), f(\overset{3}{\mathbf{B}}, \overset{2}{\mathbf{A}}, \overset{1}{\mathbf{C}})\mathfrak{P}_a(0)\big)_{\mathcal{H}}, \tag{11.4}$$

$$\|\mathfrak{P}_a(0)\| = 1, \qquad \mathbf{C}_j\mathfrak{P}_a(0) = 0, \qquad \mathbf{A}^\mu\mathfrak{P}_a(0) = a^\mu. \tag{11.5}$$

Here $(\cdot\,,\,\cdot)_{\mathcal{H}}$ denotes the inner product in a Hilbert space $\mathcal{H}$, where the operators $\mathbf{B}$, $\mathbf{A}$, and $\mathbf{C}$ act. Condition (e) from the list of Section 10 is just a simple consequence of (11.4), (11.5). The reproducing kernel $\mathcal{K}_a$ (11.3) can be written as follows:

$$\mathcal{K}_a(\bar{z}, z) = \big(\mathfrak{P}_a(0), e^{\bar{z}\mathbf{C}/\hbar}\cdot e^{z\mathbf{B}/\hbar}\mathfrak{P}_a(0)\big)_{\mathcal{H}} = \big(\mathfrak{P}_a(z), \mathfrak{P}_a(z)\big)_{\mathcal{H}}.$$

Here

$$\mathfrak{P}_a(z) \stackrel{\text{def}}{=} e^{z\mathbf{B}/\hbar}\mathfrak{P}_a(0) \tag{11.6}$$

are *coherent states* in $\mathcal{H}$ generated by the creation operators $\mathbf{B}$ and the vacuum vector $\mathfrak{P}_a(0)$ (for different definitions of coherent states see [57, 1, 59, 70, 74, 75, 83]). So, condition (f) guarantees that the norms $\big(\mathfrak{P}_a(z), \mathfrak{P}_a(z)\big)_{\mathcal{H}}$ of coherent states are positive.

Remark 11.2. Actually, condition (f) is much stronger then we really need; it is sufficient but not necessary for the positivity of norms. The *generalization of this condition is considered in details in* [55] for a wide class of products $*$ and quantum polarizations. Here we only point out that in the compact case (i.e., if symplectic leaves in $\mathcal{M}$ are compact) the infinite matrix (11.2) is restricted to a finite matrix, the sums in (11.3), (11.3a) are finite too, and the submanifold

$\mathcal{M}_0$ has to be replaced by a lattice inside of $\mathcal{M}_0$ (this lattice can be also described by the quantization condition for cohomology class of the form ω (12.2) over irreducible leaves Ω_a (13.8): $[\omega/2\pi\hbar] \in H^2(\Omega_a, \mathbb{Z})$, $a \in$ lattice $\subset \mathcal{M}_0$). In the noncompact case the assumption about nonzero convergence radius in (f), in fact, can be loosened.

Now, for each $a \in \mathcal{M}_0$, we construct a subspace E_a in the partial complexification $\mathcal{E}^{\mathbb{C}}$ of the phase space over $\mathcal{M}$. First, by $\mathcal{M}_a^{\mathbb{C}} \subset \mathcal{M}^{\mathbb{C}}$ we denote the integral leaf generated by the vector fields $\partial/\partial B_j$ ($j = 1, \dots, d$). This leaf intersects the real manifold $\mathcal{M} \subset \mathcal{M}^{\mathbb{C}}$ at the point $a \in \mathcal{M}_0$. Let E be a subbundle whose fibers consist of covectors over $\mathcal{M}^{\mathbb{C}}$ annulling $\partial/\partial A$, $\partial/\partial C$ and sufficiently small. Then the subspace E_a is obtained from the subbundle E by the restriction of its base from $\mathcal{M}^{\mathbb{C}}$ to $\mathcal{M}_a^{\mathbb{C}}$, i.e.,

$$E_a \stackrel{\text{def}}{=} E\big|_{\mathcal{M}_a^{\mathbb{C}}}.$$

We call E_a *creation submanifolds* in $\mathcal{E}^{\mathbb{C}}$; they are parameterized by points from the vacuum submanifold: $a \in \mathcal{M}_0 \subset \mathcal{M}$.

In local coordinates these creation submanifolds are described as follows:

$$E_a = \{(B, z;\ a, 0;\ 0, 0)\}.$$

Note that each E_a is a symplectic submanifold in $\mathcal{E}^{\mathbb{C}}$. The conjugate *annihilation submanifold*

$$E^a \stackrel{\text{def}}{=} \{(0, 0;\ a, 0;\ C, \bar{z})\}$$

intersects E_a at the real point $a \in \mathcal{M}_0$.

Lemma 11.1. *For each operator S of type* (10.2), *the following formulas hold*:

$$j_a(\bar{e}_z * Se_z) = \mathcal{S}_a(\hbar \overset{1}{\partial}, \overset{2}{z})\mathcal{K}_a(\bar{z}, z), \qquad j_a(S\bar{e}_z * e_z) = \mathcal{S}^a(\hbar \overset{1}{\bar{\partial}}, \overset{2}{\bar{z}})\mathcal{K}_a(\bar{z}, z),$$

where $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial\bar{z}$, and the following notation is used:

$$\mathcal{S}_a \stackrel{\text{def}}{=} \mathcal{S}\big|_{E_a}, \qquad \mathcal{S}^a \stackrel{\text{def}}{=} \mathcal{S}\big|_{E^a}. \tag{11.7}$$

Here $\mathcal{S}$ is the symbol of S in the sense of (10.2).

Proof. For brevity, we prove only the first formula of the lemma. It follows from condition (e) that $j_a(f * g) = j_a(f) \cdot j_a(g)$ if $\partial g/\partial B = 0$. Thus we have

$$\begin{aligned}
\mathcal{S}_a(\hbar \overset{1}{\partial}, \overset{2}{z})\mathcal{K}_a(\bar{z}, z) &= j_a\big(\bar{e}_z(C) * \mathcal{S}\big(\hbar \overset{1}{\partial/\partial z}, \overset{2}{z};\ a, 0;\ 0, 0\big)e_z(B)\big) \\
&= j_a\big(\bar{e}_z(C) * \mathcal{S}\big(\overset{2}{B}, \hbar \overset{1}{\partial/\partial B};\ a, 0;\ 0, 0\big)e_z(B)\big) \\
&= j_a\big(\bar{e}_z(C) * \mathcal{S}\big(\overset{2}{B}, \hbar \overset{1}{\partial/\partial B};\ A, 0;\ C, 0\big)e_z(B)\big) \\
&= j_a\big(\bar{e}_z(C) * \mathcal{S}\big(\overset{2}{B}, \hbar \overset{1}{\partial/\partial B};\ \overset{2}{A}, \hbar \overset{1}{\partial/\partial A};\ \overset{2}{C}, \hbar\overset{1}{\partial/\partial C}\big)e_z(B)\big) \\
&= j_a(\bar{e}_z * Se_z).
\end{aligned}$$

In these equalities, by B, A, C we indicate the interior variables with respect to which the $*$-product and the mapping i_a are applied. The last equality holds since the function $e_z(B)$ does not depend on A and C. The lemma is proved. □

Now let us apply Lemma 11.1 to the class of operators of left and right regular representation. Namely, denote by $\mathcal{L}_f$ and $\mathcal{R}_f$ the symbols of the operators of multiplication $f*$ and $*f$, performed in the form (10.2); that is,

$$f* = \mathcal{L}_f\big(\overset{2}{B}, \hbar\,\overset{1}{\partial/\partial B};\ \overset{2}{A}, \hbar\,\overset{1}{\partial/\partial A};\ \overset{2}{C}, \hbar\,\overset{1}{\partial/\partial C}\big),$$
$$*f = \mathcal{R}_f\big(\overset{2}{B}, \hbar\,\overset{1}{\partial/\partial B};\ \overset{2}{A}, \hbar\,\overset{1}{\partial/\partial A};\ \overset{2}{C}, \hbar\,\overset{1}{\partial/\partial C}\big).$$

Consider the restrictions of these symbols and their conjugates to the vacuum phase submanifolds E_a and E^a as in (11.7). We denote:

$$L_{f,a} \stackrel{\text{def}}{=} \overline{\mathcal{L}}^a_{\bar{f}}(\hbar\,\overset{1}{\bar{\partial}}, \overset{2}{\bar{z}}) = \mathcal{R}^a_f(\hbar\,\overset{1}{\bar{\partial}}, \overset{2}{\bar{z}}),$$
$$R_{f,a} \stackrel{\text{def}}{=} \overline{\mathcal{R}}_{\bar{f},a}(\hbar\,\overset{1}{\partial}, \overset{2}{z}) = \mathcal{L}_{f,a}(\hbar\,\overset{1}{\partial}, \overset{2}{z}). \tag{11.8}$$

Lemma 11.2. *The correspondence* $f \mapsto L_{f,a}$ *given by* (11.8) *is a homomorphism, the correspondence* $f \mapsto R_{f,a}$ *is an anti-homomorphism.*

Proof. In view of Lemma 11.1 we have:

$$L_{g,a}\mathcal{K}_a(\bar{z}, z) = R_{g,a}\mathcal{K}_a(\bar{z}, z) = j_a(\bar{e}_z * g * e_z), \tag{11.8a}$$

and hence

$$L_{f,a}L_{g,a}\mathcal{K}_a(\bar{z}, z) = j_a\big(L_{f,a}\bar{e}_z * (g * e_z)\big) = j_a\big((\bar{e}_z * f) * (g * e_z)\big).$$

The last relation here is proved in the same way as in Lemma 11.1. Thus, we obtain

$$L_{f,a}L_{g,a}\mathcal{K}_a(\bar{z}, z) = j_a\big(\bar{e}_z * (f * g) * e_z\big) = L_{f*g,a}\mathcal{K}_a(\bar{z}, z).$$

Similar derivations show that $R_{f,a}R_{g,a} = R_{g*f,a}$. The lemma is proved. □

Thus, we obtain the following theorem.

Theorem 11.1. (i) *Let the normal star-product over* $\mathcal{M}$ *admit a vacuum. Let* $\mathcal{M}_0$ *be a vacuum submanifold in* $\mathcal{M}$. *Then for any* $a \in \mathcal{M}_0$, *there is a homomorphism of the normal star-product algebra over* $\mathcal{M}$ *to the algebra of antiholomorphic differential operators. On the coordinate functions this homomorphism acts by the following formulas*

$$C \mapsto L_{C,a} \stackrel{\text{def}}{=} \overline{\mathcal{L}}^a_B\big(\hbar\,\overset{1}{\bar{\partial}}, \overset{2}{\bar{z}}\big) = \hbar\bar{\partial},$$
$$B \mapsto L_{B,a} \stackrel{\text{def}}{=} \overline{\mathcal{L}}^a_C\big(\hbar\,\overset{1}{\bar{\partial}}, \overset{2}{\bar{z}}\big), \tag{11.9}$$
$$A \mapsto L_{A,a} \stackrel{\text{def}}{=} \overline{\mathcal{L}}^a_A\big(\hbar\,\overset{1}{\bar{\partial}}, \overset{2}{\bar{z}}\big).$$

Here the symbols $\mathcal{L}^a_B$, $\mathcal{L}^a_A$, $\mathcal{L}^a_C$ *are determined from symbols of the left regular representation* (*Theorem* 10.1) *by the restriction procedure* (11.7) *onto the creation submanifold* E^a. *The general formula for this homomorphism is* (11.8), *or the following*:

$$f \mapsto L_{f,a} = f(\overset{3}{L}_{B,a}, \overset{2}{L}_{A,a}, \overset{1}{L}_{C,a}). \tag{11.10}$$

The representation (11.10) *of the normal star-product algebra is irreducible and Hermitian in the Hilbert space* $\mathcal{P}_a$ *of antiholomorphic distributions with the Hilbert structure* (11.3a), *that is,* $(L_{A,a})^* = L_{A,a}$ *and* $(L_{B,a})^* = L_{C,a}$ *in* $\mathcal{P}_a$. *The vacuum vector in* $\mathcal{P}_a$ *is the unity function* $1 \equiv 1(\bar{z})$, *and so,* $L_{C,a}1 = 0$, $L_{A,a}1 = a$.

(ii) *All statements in* (i) *remain valid after the following change: homomorphism* $\to$ *anti-homomorphism, anti-holomorphic* $\to$ *holomorphic,* $L_{f,a} \to R_{f,a}$, *left* $\to$ *right,* $\mathcal{L}^a \to \mathcal{R}_a$, *creation* $\to$ *annihilation, and* $\mathcal{P}_a \to \overline{\mathcal{P}}_a$. *After this change, the analog of formula* (11.10) *looks like*

$$f \mapsto R_{f,a} = f\big(\overset{1}{R}_{B,a}, \overset{2}{R}_{A,a}, \overset{3}{R}_{C,a}\big). \tag{11.10a}$$

Remark 11.3. Actually, an interesting point of Theorem 11.1 is that the irreducible representations of the star-product are constructed via symbols of operators of the left or right regular representation. These symbols are evaluated explicitly starting from a quantum $\star$-product over $\mathcal{M}$; moreover, these symbols are constructed geometrically; for instance, the leading symbols determine the groupoid reduction mappings in (10.3). Note that to obtain irreducible representations, as we see, it is not sufficient to have a star-product over $\mathcal{M}$ defined only as a formal series in $\hbar$; it must be first extended to the phase space level (9.2a) or (10.1).

Another interesting point is that in (11.9), (11.10), (11.10a) we use only the restriction (11.7) of all symbols to the creation or annihilation submanifolds E^a or E_a.

Remark 11.4. As in Remark 11.1, let $\mathcal{H}$ be an abstract Hilbert space of a representation of the algebra $\mathcal{F}_\hbar(\mathcal{M})$:

$$f \mapsto \hat{f} = f(\overset{3}{\mathbf{B}}, \overset{2}{\mathbf{A}}, \overset{1}{\mathbf{C}}). \tag{11.11}$$

For each $a \in \mathcal{M}_0$ the *coherent transform*

$$\mathfrak{p}_a : \mathcal{P}_a \to \mathcal{H} \tag{11.12}$$

can be defined by the relation

$$\big(u, \mathfrak{p}_a(g)\big)_{\mathcal{H}} = \big((\mathfrak{P}_a, u)_{\mathcal{H}}, g\big)_{\mathcal{P}_a}, \qquad \forall g \in \mathcal{P}_a, \quad u \in \mathcal{H}. \tag{11.13}$$

The image of $\mathfrak{p}_a$ is a subspace $\mathcal{H}_a \subset \mathcal{H}$ that coincides with the closure of the linear envelope of the set of coherent state, i.e., $\mathcal{H}_a = \operatorname{span}\{\mathfrak{P}_a(z)\}$. The mapping $\mathfrak{p}_a : \mathcal{P}_a \to \mathcal{H}_a$ is unitary, and its inverse is given by $\mathfrak{p}_a^{-1}(u) = (\mathfrak{P}_a, u)_{\mathcal{H}}$. The Hilbert subspace $\mathcal{H}_a$ is the space, where the irreducible component of the representation (11.11) acts. Under the coherent transform (11.12) this component turns into (11.10).

From (11.6) it readily follows that the coherent states satisfy the equations

$$\begin{aligned} &\mathbf{B}\mathfrak{P}_a(z) = \mathcal{L}_{B,a}(\overset{1}{\hbar\,\partial}, \overset{2}{z})\mathfrak{P}_a(z), \qquad \mathbf{C}\mathfrak{P}_a(z) = \mathcal{L}_{C,a}(\overset{1}{\hbar\,\partial}, \overset{2}{z})\mathfrak{P}_a(z), \\ &\mathbf{A}\mathfrak{P}_a(z) = \mathcal{L}_{A,a}(\overset{1}{\hbar\,\partial}, \overset{2}{z})\mathfrak{P}_a(z). \end{aligned} \tag{11.14}$$

We recall that $\mathcal{L}_{B,a}$, $\mathcal{L}_{C,a}$, $\mathcal{L}_{A,a}$ are restrictions on the annihilation submanifold E_a of symbols of the left regular representation operators $B*$, $C*$, $A*$. The system (11.14) could be considered as a system of equations determining coherent states.

Similar equations hold for the reproducing kernel $\mathcal{K}_a$. Say, from (11.8a) and (11.9) we obtain the following statement.

In the particular case in which $\mathfrak{P}_a = \mathcal{K}_a$, the operators **B**, **C**, **A** are given by (11.9); so, $\mathbf{C} = \hbar\bar{\partial}$, and (11.14) implies a system of equations determining the function $\mathcal{K}_a$.

Corollary 11.1. *The reproducing kernel satisfies the equations*

$$\hbar\bar{\partial}\mathcal{K}_a = \mathcal{L}_{C,a}(\hbar \overset{1}{\partial}, \overset{2}{z})\mathcal{K}_a, \qquad \mathcal{K}_a\big|_{z=0} = 1, \tag{11.15}$$

and

$$\operatorname{Im}(\mathcal{L}_{A,a}(\hbar \overset{1}{\partial}, \overset{2}{z})\mathcal{K}_a) = 0. \tag{11.15a}$$

In particular, if one represents the reproducing kernel in the exponential form

$$\mathcal{K}_a = e^{F_a/\hbar}, \tag{11.16}$$

then for the quantum Kählerian potential $F_a = F_a(\bar{z}, z)$ the following quantum Hamilton–Jacobi equation holds:

$$\bar{\partial} F_a = \mathcal{L}_{C,a}(\partial F_a \overset{1}{+} \hbar\partial, \overset{2}{z})\, 1, \qquad F_a\big|_{\bar{z}=0} = 0, \tag{11.17}$$

together with the additional condition

$$\operatorname{Im}\big(\mathcal{L}_{A,a}(\partial F_a \overset{1}{+} \hbar\partial, \overset{2}{z})\, 1\big) = 0. \tag{11.17a}$$

Note that in (11.15), (11.15a), (11.17), (11.17a) we have not single equations, but systems of equations, since $z = (z^1, \dots, z^d)$, $\partial = (\partial/\partial z^1, \dots, \partial/\partial z^d)$, and $\mathcal{L}_{C,a}$ is a vector-valued function: $\mathcal{L}_{C,a}(B, z) = (\mathcal{L}_{C_{1,a}}(B, z), \dots, \mathcal{L}_{C_{d,a}}(B, z))$. The components of $\mathcal{L}_{C,a}$ quantum commute:

$$\mathcal{L}_{C_{s,a}}(\hbar \overset{1}{\partial} + B, \overset{2}{z})\mathcal{L}_{C_{j,a}}(B, z) = \mathcal{L}_{C_{j,a}}(\hbar \overset{1}{\partial} + B, \overset{2}{z})\mathcal{L}_{C_{s,a}}(B, z) \qquad \forall s, j = 1, \dots, d,$$

what follows from the commutativity $C_s * C_j = C_j * C_s$ (see condition (c) in Section 10).

System (11.15) allow us to derive the kernel $\mathcal{K}_a$ or the potential F_a directly and *independently from the definition via the power series* (11.3). In particular, (11.7), (11.7a) can be readily solved by formal $\hbar$-expansion

$$F_a = F_a^{(0)} + \hbar F_a^{(1)} + \hbar^2 F_a^{(2)} + \cdots. \tag{11.18}$$

Equations for the leading part $F_a^{(0)}$ see in (15.1); geometry of higher terms is considered in [45–47, 55].

Also note that from (11.15) we obtain

$$\mathcal{K}_a(\bar{z}, z) = \exp\left\{\frac{z}{\hbar} L_{B,a}\right\} 1(\bar{z}),$$

and so, the reproducing kernel has the form (11.6) in the Hilbert space $\mathcal{H}_a = \mathcal{P}_a$ with the vacuum 1 and with the creation operators $L_{B,a}$. For each fixed z, let us denote

$$\pi(z) \overset{\text{def}}{=} \mathcal{K}_a(\cdot, z) \in \mathcal{P}_a. \tag{11.19}$$

Thus, in the case $\mathcal{H}_a = \mathcal{P}_a$, the functions $\pi(z) \equiv \mathfrak{P}_a(z)$ are coherent states in the space $\mathcal{P}_a$, and the coherent transform (11.12) is just the identical mapping $\mathfrak{p}_a = I$.

12. Explicit power series for the Wick product over Kähler manifolds

From Theorem 11.1, for each $a \in \mathcal{M}_0$ we know the irreducible representation of the $*$-product algebra $\mathcal{F}_\hbar(\mathcal{M})$ by antiholomorphic operators (11.10). Now, taking the natural next step, we would like to realize these operators by functions (symbols) with some new star-product $\underset{a}{*}$ which is nondegenerate, i.e., has the trivial center. Further, this new function algebra will be identified with a space of functions over a submanifold $\Omega_a \subset \mathcal{M}$.

The problem is to dequantize the representation (11.10), preserving the triviality of the center, or, in other words, preserving the irreducibility.

In this section we describe (and clarify) the procedure of the Wick–Klauder–Berezin [5–9, 57, 58] quantization/dequantization.

First of all, over the space of functions in $\bar{z}, z \in \mathbb{C}^d$, we now introduce a star-product $\underset{a}{*}$ generated by a reproducing kernel. For simplicity, we denote the reproducing kernel by $\mathcal{K}_a$ as previously, although now the kernel could be a general function, i.e., it is not necessarily given by (11.3). Anyway, this function is an integral kernel of the unity operator in a Hilbert space $\mathcal{P}_a$ of anti-holomorphic distributions.

First, note that the Hilbert structure naturally generates the "matrix" multiplication of functions

$$(\varphi \times \psi)(\bar{z}, z) \stackrel{\text{def}}{=} \big(\overline{\varphi(\bar{z}, \cdot)}, \psi(\cdot, z)\big)_{\mathcal{P}_a}.$$

Here the scalar product in $\mathcal{P}_a$ is considered with respect to the arguments marked by dots. The unity element for this "matrix" multiplication is the function $\mathcal{K}_a$.

Now, one can introduce a new multiplication [5, 6, 57, 81]

$$\varphi \underset{a}{*} \psi \stackrel{\text{def}}{=} \frac{1}{\mathcal{K}_a}\big((\mathcal{K}_a\varphi) \times (\mathcal{K}_a\psi)\big). \tag{12.1}$$

The unity element for this multiplication is 1. The algebra of functions with product (12.1) will be denoted by $\mathcal{F}_\hbar^a$.

Note that there is a simple geometric interpretation of products like (12.1) via integrals of the closed form

$$\omega = i\bar{\partial}_j\partial_k F_a \, d\bar{z}^j \wedge dz^k, \qquad F_a = \ln \mathcal{K}_a, \tag{12.2}$$

over certain *complex membranes* [45–50]. In fact, this form is the only object which is needed to determine the product $\underset{a}{*}$ (see also in [40]). In this section we derive explicit formulas connecting ω and $\underset{a}{*}$.

Note that all the key properties, like the properties (a)–(e) from the list of Section 10, hold for the product (12.1):

$$\begin{aligned} &\overline{\varphi \underset{a}{*} \psi} = \bar{\psi} \underset{a}{*} \bar{\varphi}; \\ &\varphi \underset{a}{*} \psi - \psi \underset{a}{*} \varphi = -i\hbar\{\varphi, \psi\} + O(\hbar^2); \\ &\varphi \underset{a}{*} \psi = \varphi\psi \qquad \text{if } \psi \text{ is holomorphic}, \end{aligned} \tag{12.3}$$

where $\{\cdot\,,\,\cdot\}$ are the Poisson brackets generated by the symplectic form

$$\omega_0 \stackrel{\text{def}}{=} i\bar{\partial}\partial F_a^{(0)} \, d\bar{z} \wedge dz, \qquad \omega_0 = \omega\big|_{\hbar=0}. \tag{12.2a}$$

The first and the last properties in (12.3) can readily be verified by using the definition (12.1). But the second one (considered only on functions, which are regular as $\hbar \to 0$) is not so simple to prove. In order to prove this second property, we use and develop now the Fock–Dirac idea of creation-annihilation operators. This provides us not only with (12.3) but with explicit formulas for all terms of power $\hbar$-expansions of the product $\underset{a}{*}$ in terms of the Kählerian form (12.2) exclusively.

In our Hilbert space $\mathcal{P}_a$ (11.3a) let us consider the operators of multiplication by coordinate functions $\bar{z}^1, \dots, \bar{z}^d$, and introduce the adjoint operators

$$\hat{z}^s \overset{\text{def}}{=} (\bar{z}^s)^*, \qquad s = 1, \dots, d. \tag{12.4}$$

Since $\mathcal{K}_a$ is the integral kernel of the unity operator, for each fixed z the function $\pi(z)$ (11.19) is the eigenfunction for all operators $\hat{z}^s$:

$$\hat{z}^s \pi(z) = z^s \cdot \pi(z), \qquad s = 1, \dots, d. \tag{12.5}$$

In particular,

$$\hat{z}^s\, 1 = 0, \qquad s = 1, \dots, d. \tag{12.5a}$$

In view of (12.4), (12.5a), the operators $\hat{z}^s$ could be considered as *annihilation operators* in $\mathcal{P}_a$ with the vacuum vector $1 \in \mathcal{P}_a$.

To each functions $\varphi = \varphi(\bar{z}, z) \in \mathcal{F}^a_\hbar$, we assign an operator $\hat{\varphi}$ in the Hilbert space $\mathcal{P}_a$ by the formula

$$\hat{\varphi} \overset{\text{def}}{=} \varphi(\overset{2}{\bar{z}}, \overset{1}{\hat{z}}). \tag{12.6}$$

In view of (12.5), this formula explicitly reads

$$\hat{\varphi}\big(g(\bar{z})\big) = \big(\overline{\varphi(\bar{z}, \cdot)}\pi(z), g\big)_{\mathcal{P}_a}, \qquad g \in \mathcal{P}_a. \tag{12.6a}$$

The next statement concerns the key properties of quantization (12.6).

Theorem 12.1. *The following equalities hold:*

$$\hat{\varphi}_1 \hat{\varphi}_2 = \widehat{\varphi_1 \underset{a}{*} \varphi_2}, \tag{12.7}$$

$$\varphi \underset{a}{*} = \frac{1}{\mathcal{K}_a} \circ \hat{\varphi} \circ \mathcal{K}_a. \tag{12.8}$$

In particular, if $\varphi = \bar{\partial} F_a$, *then* (12.6a) *implies*

$$\widehat{\bar{\partial} F_a} = \hbar \bar{\partial}. \tag{12.9}$$

Proof. Equalities (12.7), (12.8) follow directly from (12.6a) and (12.1). From (12.6a) we also have

$$\widehat{\bar{\partial} F_a}\, g(\bar{z}) = \big(\partial F_a(\cdot, z)\mathcal{K}_a(\cdot, z), g\big)_{\mathcal{P}_a} = \hbar\bar{\partial}\big(\mathcal{K}_a(\cdot, z), g\big)_{\mathcal{P}_a} = \hbar\bar{\partial} g(\bar{z}),$$

since $\mathcal{K}_a$ is the integral kernel of the unity operator. □

An explicit form of (12.9) is

$$\bar{\partial}_s F_a(\overset{2}{\bar{z}}, \overset{1}{\hat{z}}) = \hbar \bar{\partial}_s, \qquad s = 1, \dots, d. \tag{12.9a}$$

Note that *from this equation one can derive the annihilation operator $\hat{z}$ explicitly as a function in $\bar{z}$ and $\bar{\partial}$* while the matrix

$$\omega_{\ell s} \stackrel{\text{def}}{=} \partial_\ell \bar{\partial}_s F_a \tag{12.10}$$

is not degenerate. We are not distracted for this calculation. Let us now derive an explicit formula for the quantum $\underset{a}{*}$-tensor from (12.9a).

Following the general definition (2.4), in order to determine the quantum $\underset{a}{*}$-tensor in our case, we have to perform the commutators $[\hat{z}^r, \bar{z}^\ell]$ via operators of type (12.6), i.e.,

$$[\hat{z}^r, \bar{z}^\ell] = \hbar N^{\ell r}(\overset{2}{\bar{z}}, \overset{1}{\hat{z}}). \tag{12.11}$$

The matrix $N = ((N^{\ell r}))$ represents the nonzero part of the quantum $\underset{a}{*}$-tensor.

At the same time we introduce analogs of the operators $\mathrm{ad}^j_\hbar$ from Section 3; let us denote

$$[\hat{z}^r, \hat{\varphi}] \stackrel{\text{def}}{=} \hbar \widehat{\bar{D}^r \varphi}, \qquad [\hat{\varphi}, \bar{z}^\ell] \stackrel{\text{def}}{=} \hbar \widehat{D^\ell \varphi}. \tag{12.12}$$

The operators $\bar{D}^r$, D^ℓ are Euler–Poisson quantum vector fields with respect to the product $\underset{a}{*}$. They can be derived via the quantum $\underset{a}{*}$-tensor N using the general commutation formulas (see, for example, in [54]). Say, the formula for D^ℓ is the following

$$\begin{aligned}
\widehat{D^\ell \varphi} &= \int_0^1 \overset{2}{\widehat{N^{\ell r}}}\, \partial_r \varphi\big(\overset{4}{\bar{z}}, \tau \overset{3}{\hat{z}} + (1-\tau) \overset{2}{\hat{z}}\big)\, d\tau \\
&= \sum_{|\alpha|\geqslant 0} \frac{1}{(|\alpha|+1)\alpha!} \underbrace{[\hat{z}, \dots, [\overset{2}{\hat{z}}}_{\alpha}, \widehat{N^{\ell r}}] \dots] \partial^\alpha \partial_r \varphi(\overset{3}{\bar{z}}, \overset{1}{\hat{z}}).
\end{aligned}$$

Here the multi-commutator can be determined from the definition (12.12) as follows: $\underbrace{[\hat{z}, \dots, [\hat{z}}_{\alpha}, \widehat{N^{\ell r}}] \dots] = \hbar^{|\alpha|} \widehat{\bar{D}^\alpha N^{\ell r}}$. Thus we obtain

$$D^\ell = \sum_{|\alpha|\geqslant 0} \frac{\hbar^{|\alpha|}}{(|\alpha|+1)\alpha!} \bar{D}^\alpha(N^{\ell r})\, \partial^\alpha \partial_r = N^{\ell r}\, \mathbf{p}(\hbar \underset{\leftarrow}{\bar{D}} \cdot \underset{\rightarrow}{\partial})\, \partial_r, \tag{12.13}$$

where $\mathbf{p}(x) = (e^x - 1)/x$ (compare with (3.2), (4.2), and (6.5)).

For the conjugate operators $\bar{D}^\ell$, an analog of (12.13) looks as follows:

$$\bar{D}^\ell = \sum_{|\alpha|\geqslant 0} \frac{\hbar^{|\alpha|}}{(|\alpha|+1)\alpha!} D^\alpha(N^{r\ell})\, \bar{\partial}^\alpha\, \bar{\partial}_r = N^{r\ell}\, \mathbf{p}(\hbar \underset{\leftarrow}{D} \cdot \underset{\rightarrow}{\bar{\partial}})\, \bar{\partial}_r, \tag{12.14}$$

From the pair of equations (12.13), (12.14) we readily calculate the $\hbar$-expansion for both D and $\bar{D}$. The first three terms are the following:

$$\begin{aligned}
D^\ell &= N^{\ell s}\partial_s + \frac{\hbar}{2}(N^{pq}\bar{\partial}_p N^{\ell s})\partial_q \partial_s \\
&\quad + \hbar^2\big(\tfrac{1}{4}(N^{rm}\partial_m N^{pq}\bar{\partial}_r \bar{\partial}_p N^{\ell s})\partial_q \partial_s + \tfrac{1}{6}(N^{rm}\bar{\partial}_r(N^{pq}\bar{\partial}_p N^{\ell s}))\partial_m \partial_q \partial_s\big) \\
&\quad + \cdots .
\end{aligned} \tag{12.15}$$

The formula for $\bar{D}^\ell$ differs from (12.15) just by the replacing $\partial \to \bar{\partial}$ and $N^{ij} \to N^{ji}$.

Now let us consider the operators of the left and right regular representation

$$\hat{z}\hat{\varphi} \overset{\text{def}}{=} \widehat{L\varphi}, \qquad \hat{\varphi}\hat{\bar{z}} \overset{\text{def}}{=} \widehat{R\varphi}.$$

Then

$$L^r = z^r + \hbar \overline{D}^r, \qquad R^\ell = \bar{z}^\ell + \hbar D^\ell, \qquad r, \ell = 1, \dots, d.$$

Since $[L^r, z^s] = [R^\ell, \bar{z}^s] = 0$, we have

$$[\overline{D}^r, z^s] = [D^\ell, \bar{z}^s] = 0, \qquad \forall s, r, \ell.$$

Thus, the product $\underset{a}{*}$, which is the product of normal symbols of operators (12.6), can be derived as follows:

$$\begin{aligned} \varphi \underset{a}{*} \psi &= \varphi(\overset{2}{\bar{z}}, \overset{1}{L})\psi = \varphi(\overset{2}{\bar{z}}, \overset{1}{z + \hbar \overline{D}})\psi = \sum_{|\alpha| \geqslant 0} \frac{\hbar^{|\alpha|}}{\alpha!} \bar{\partial}^\alpha \varphi \cdot \overline{D}^\alpha \psi \\ &= \varphi \exp\{\hbar \underset{\leftarrow}{\bar{\partial}} \cdot \underset{\rightarrow}{\overline{D}}\}\, \psi, \end{aligned} \tag{12.16}$$

or, simultaneously, as follows

$$\begin{aligned} \varphi \underset{a}{*} \psi &= \psi(\overset{1}{R}, \overset{2}{z})\varphi = \psi(\overset{1}{\bar{z} + \hbar D}, \overset{2}{z})\varphi = \sum_{|\alpha| \geqslant 0} \frac{\hbar^{|\alpha|}}{\alpha!} \bar{\partial}^\alpha \psi \cdot D^\alpha \varphi \\ &= \varphi \exp\{\hbar \underset{\leftarrow}{D} \cdot \underset{\rightarrow}{\bar{\partial}}\}\, \psi, \end{aligned} \tag{12.16a}$$

The leading terms of $\hbar$-expansion for the $\underset{a}{*}$-product are

$$\begin{aligned} \varphi \underset{a}{*} \psi &= \varphi \cdot \psi + \hbar N^{rs} \partial_s \varphi \cdot \bar{\partial}_r \psi \\ &\quad + \frac{\hbar^2}{2} \big(N^{pq} \bar{\partial}_p N^{rs} \partial_q \partial_s \varphi \cdot \bar{\partial}_r \psi + N^{rs} \partial_s (N^{pq} \partial_q \varphi) \cdot \bar{\partial}_p \bar{\partial}_r \psi\big) + \cdots . \end{aligned} \tag{12.16b}$$

If the quantum tensor N in (12.11) is given *a priori*, then formulas for the star-product (12.16), (12.16a), (12.16b) are final. But, on the other hand, the quantum tensor N itself could be determined, say, via the form (12.2)

$$\omega = i \omega_{\ell s}\, d\bar{z}^s \wedge dz^\ell.$$

For this purpose, let us calculate the commutator with $\bar{z}^\ell$ on both sides of (12.9a):

$$D^\ell(\bar{\partial}_s F_a) = \delta^\ell_s, \qquad s, \ell = 1, \dots, d.$$

On the right δ^ℓ_s denotes the Kronecker symbol. After the substitution of (12.13) and (12.10), we obtain

$$N^{\ell r}\, \mathbf{p}(\hbar \underset{\leftarrow}{\overline{D}} \cdot \underset{\rightarrow}{\partial}) \omega_{rs} = \delta^\ell_s \tag{12.17}$$

or, if we use the notation of Section 3,

$$N^{\ell r} \odot \omega_{rs} = \delta^\ell_s. \tag{12.17a}$$

From this relation we can readily derive the quantum tensor N via the Kählerian form ω. Let us denote by

$$g^{\ell s} = (\omega^{-1})^{\ell s}$$

the elements of the inverse matrix $((\omega_{\ell s}))^{-1}$. Then from (12.17) the leading terms of $\hbar$-expansion for the quantum tensor N are the following:

$$
\begin{aligned}
N^{\ell r} = g^{\ell r} &+ \tfrac{\hbar}{2}\partial_m g^{pr}\bar{\partial}_p g^{\ell m} - \hbar^2\Big[\tfrac{1}{4}g^{sr}g^{jk}(\partial_k g^{pi})(\bar{\partial}_p\bar{\partial}_j g^{\ell m})\partial_i\omega_{ms} \\
&+ \tfrac{1}{6}g^{sr}g^{jk}\bar{\partial}_j(g^{pq}\bar{\partial}_p g^{\ell m})\partial_q\partial_k\omega_{ms} + \tfrac{1}{4}g^{sr}(\partial_q g^{ji})(\bar{\partial}_j g^{pq})(\bar{\partial}_p g^{\ell m})\partial_i\omega_{ms} \\
&- \tfrac{1}{4}(\partial_i g^{pr})\bar{\partial}_p\big((\partial_m g^{ji})(\bar{\partial}_j g^{\ell m})\big)\Big] + O(\hbar^3).
\end{aligned}
\tag{12.18}
$$

After the substitution of this expansion into (12.16b) we obtain an explicit $\hbar$-expansion for the $\underset{a}{*}$-product

$$
\begin{aligned}
\varphi \underset{a}{*} \psi = \varphi\cdot\psi &+ \hbar g^{rs}\partial_s\varphi\cdot\bar{\partial}_r\psi \\
&+ \frac{\hbar^2}{2}\Big[(g^{pq}\bar{\partial}_p g^{rs})\partial_q\partial_s\varphi\cdot\bar{\partial}_r\psi + g^{rs}\partial_s(g^{pq}\partial_q\varphi)\cdot\bar{\partial}_p\bar{\partial}_r\psi \\
&\qquad + (\partial_m g^{ps})(\bar{\partial}_p g^{rm})\partial_s\varphi\cdot\bar{\partial}_r\psi\Big] + O(\hbar^3).
\end{aligned}
\tag{12.19}
$$

As a simple consequence from this expansion we prove the second relation in (12.3) with the Poisson brackets given by

$$
\{\varphi, \psi\} = i g_0^{rs}\big(\partial_s\varphi\bar{\partial}_r\psi - \partial_s\psi\bar{\partial}_r\varphi\big),
\tag{12.20}
$$

where $g_0 = g\big|_{\hbar=0}$ is the metric corresponding to the classical symplectic form (12.2a).

Operators D^ℓ can also be represented via the metric from expansions (12.15) and (12.18):

$$
D^\ell = g^{\ell s}\partial_s + \frac{\hbar}{2}\Big[g^{pq}(\bar{\partial}_p g^{\ell s})\partial_q\partial_s + (\partial_m g^{ps})(\bar{\partial}_p g^{\ell m})\partial_s\Big] + O(\hbar^2).
\tag{12.21}
$$

Note that although all calculations above were performed in a local chart, the resulting expansion (12.19) is invariant with respect to a change of local coordinates $\bar{z}$, z. Thus, we obtain the following statement.

Theorem 12.2. *Any (pseudo-) Kählerian 2-form ω on a manifold Ω generates the associative star-product over Ω with properties* (12.3), (12.1) *(the function $\mathcal{K}_a$ in* (12.1) *is given by* (11.16), *and F_a is the local potential of the form ω* (12.2)). *This product can be calculated by formula* (12.16) *or* (12.16a), *where the operators D, $\bar{D}$ are derived from* (12.13), (12.14) *(and the quantum tensor N is taken from* (12.17a)). *Several terms of $\hbar$-expansions for the star-product, for D, and for N are given in* (12.19), (12.21) *and* (12.18). *All terms of expansion* (12.19) *are differential invariants on Ω.*

Thus we have obtained remarkable differential invariants of higher orders $2, 3, \ldots$ on an arbitrary (pseudo-) Kähler manifold. They are just coefficients at powers of $\hbar$ in the expansion

$$
\varphi(z) \underset{a}{*} \psi(\bar{z}) = \varphi(z)\psi(\bar{z}) + \sum_{m=1}^{\infty}\frac{\hbar^m}{m!}I_m\big(\varphi(z)\psi(\bar{z})\big).
\tag{12.22}
$$

As it follows from (12.19), the first invariant I_1 is the half of the Laplace–Beltrami operator

$$
I_1 = g^{rs}\partial_s\bar{\partial}_r = \tfrac{1}{2}\Delta.
$$

The next invariant I_2 one can see in (2.19) as the coefficient at $\hbar^2$:

$$I_2 = \tfrac{1}{4}\Delta^2 + \Big[(\bar\partial_r g^{ps})(\partial_s g^{rq}) - g^{rs}\partial_s\bar\partial_r g^{pq}\Big]\partial_q\bar\partial_p.$$

Here the fourth order summand is just the square of the Laplace–Beltrami operator, but we also obtain an additional second order differential invariant determined by the following tensor of $(1,1)$-type:

$$\widetilde{g}^{pq} = (\bar\partial_r g^{ps})(\partial_s g^{rq}) - g^{rs}\partial_s\bar\partial_r g^{pq}. \tag{12.23}$$

There are many others in the next terms of (12.19). The above procedure guarantees an explicit derivation of all invariants I_m (see also [46], in particular, Remarks 1.1 and 1.2 therein).

Remark 12.1. Formula (12.6) and Theorem 12.1 provide a representation of the star-product algebra (12.16) by anti-holomorphic operators. The construction of normal ordering functions in creation-annihilation operators of type (12.6) is called the Wick quantization [5]. Normal symbol φ of the operator $\hat\varphi$ (12.6) is called the *Wick symbol*. That is why we call the product $\underset{a}{*}$ the *Wick product*. About other ways to define the Wick product, and for the discussion around this topic see in [6, 8, 12, 14, 36, 37, 40, 46, 48–50, 57, 58, 68, 79, 81, 85].

The Wick symbol φ of an operator $\hat\varphi$ in $\mathcal{P}_a$ can be reconstructed by the formula

$$\varphi(\bar z, z) = \frac{\big(\pi(z), \hat\varphi\pi(z)\big)_{\mathcal{P}_a}}{\big(\pi(z), \pi(z)\big)_{\mathcal{P}_a}}.$$

This is the *dequantization formula* due to Berezin and Klauder in terms of coherent states in the Hilbert space $\mathcal{P}_a$. Of course, this formula can be exploited for operators acting in an abstract Hilbert space $\mathcal{H}$. One just has to replace $\pi(z)$ by the abstract coherent state $\mathfrak{P}_a(z)$ (11.6). The Wick symbol φ of an abstract operator Φ, acting in $\mathcal{H}$, is determined as follows:

$$\varphi(\bar z, z) = \frac{\big(\mathfrak{P}_a(z), \Phi\mathfrak{P}_a(z)\big)_{\mathcal{H}}}{\big(\mathfrak{P}_a(z), \mathfrak{P}_a(z)\big)_{\mathcal{H}}}, \qquad \Phi \equiv \hat\varphi. \tag{12.24}$$

If we quantize in this way the coordinate function $\bar z \to \hat{\bar z}$ and denote $\hat z \overset{\text{def}}{=} (\hat{\bar z})^*$, then the operators $\hat{\bar z}$, $\hat z$ possess all properties of creations-annihilations in the Hilbert space $\mathcal{H}$:

$$\hat z\mathfrak{P}_a(z) = z\mathfrak{P}_a(z), \qquad \hat z\mathfrak{P}_a(0) = 0,$$
$$[\hat z, \hat{\bar z}] = \hbar N(\overset{2}{\hat{\bar z}}, \overset{1}{\hat z}), \qquad \hat\varphi = \varphi(\overset{2}{\hat{\bar z}}, \overset{1}{\hat z}).$$

Now let us recall the *anti-Wick quantization* given by the anti-normal ordering

$$\psi(\bar z, z) \mapsto \underset{\wedge}{\psi} \overset{\text{def}}{=} \psi(\overset{1}{\hat{\bar z}}, \overset{2}{\hat z}). \tag{12.25}$$

One can readily prove that

$$\underset{\wedge}{\psi} = \frac{1}{(2\pi\hbar)^d}\int_\Omega \psi(\bar z, z)\Pi(\bar z, z)\, dm(\bar z, z). \tag{12.25a}$$

Here $\Pi(\bar z, z)$ is the orthogonal projection in $\mathcal{P}_a$ onto the linear subspace generated by the function

$\pi(z)$ (11.19), and dm is the *reproducing measure* taking part in the resolutions of unity:

$$\begin{aligned} &\frac{1}{(2\pi\hbar)^d}\int_\Omega \Pi(\bar z, z)\, dm(\bar z, z) = I, \\ &\frac{1}{(2\pi\hbar)^d}\int_\Omega p(\overline{w}, w; \bar z, z)\, dm(\bar z, z) = 1, \qquad \forall \overline{w}, w, \end{aligned} \tag{12.26}$$

where

$$p(\overline{w}, w; \bar z, z) \stackrel{\text{def}}{=} \frac{\mathcal{K}_a(\overline{w}, z)\mathcal{K}_a(\bar z, w)}{\mathcal{K}_a(\overline{w}, w)\mathcal{K}_a(\bar z, z)}, \qquad 2d = \dim\Omega. \tag{12.26a}$$

See in [45–50] for geometric interpretation of this and other interesting equalities via integrals over complex membranes of different configurations; after this interpretation, properties like (12.26) seem, in fact, very similar to the Duistermaat–Heckman formulas.

Note also that the reproducing mesure dm is related to the Hilbert structure in the space $\mathcal{P}_a$ (11.3a) as follows:

$$\|g\|^2_{\mathcal{P}_a} = \int_\Omega |g|^2 e^{-F_a/\hbar}\, dm.$$

The function ψ in (12.25), (12.25a) is called the *anti-Wick symbol* of the operator $\underset{\wedge}{\psi}$. The product of these symbols also has a geometric interpretation via membranes, as well as the transformation between Wick and anti-Wick symbols. Note that in terms of the differential invariants I_m obtained above, the transformation from anti-Wick to Wick symbols is given by the formula

$$\underset{\wedge}{\psi} = \hat\varphi, \qquad \varphi = \Big(I + \sum_{m=1}^{\infty}\frac{\hbar^m}{m!} I_m\Big)\psi. \tag{12.27}$$

In the particular case of symplectic leaves in semi-simple Lie algebras, this transformation was studied in [8], and invariants I_m were calculated in terms of Casimir elements of the Lie algebra.

As we see from (12.22), the operator

$$P = I + \sum_{m=1}^{\infty}\frac{\hbar^m}{m!} I_m = \exp\big\{\hbar \overset{2}{D}\cdot\overset{1}{\bar\partial}\,\big\} = \exp\big\{\hbar \overset{2}{\bar D}\cdot\overset{1}{\partial}\,\big\} \tag{12.28}$$

completely determines the Wick product, as well as the transfer to the anti-Wick product (12.27): $\varphi = P\psi$. In [45] the operator P was called the *probability operator*; its integral kernel is the probability function p (12.26a) which can geometrically be interpreted via complex membranes or via the Calabi diastasis function [16]; see details in [14, 46–50, 82].

Formula (12.28) implies that the probability operator P is formally invertible and the inverse operator can readily be calculated as the $\hbar$-series

$$P^{-1} = \big(\exp\big\{\hbar \overset{2}{\bar D}\cdot\overset{1}{\partial}\,\big\}\big)^{-1} = I - \frac{\hbar}{2}\Delta - \frac{\hbar^2}{2}\big(\widetilde{g}^{pq}\bar\partial_p\partial_q - \tfrac14\Delta^2\big) + O(\hbar^3). \tag{12.29}$$

Here the tensor $\widetilde{g}$ is given by (12.23). Now we denote

$$Q \stackrel{\text{def}}{=} \frac{1}{\hbar^2}\Big(I - \frac{\hbar}{2}\Delta - P^{-1}\Big) = \widetilde{g}^{pq}\bar\partial_p\partial_q - \tfrac14\Delta^2 + O(\hbar). \tag{12.30}$$

We also introduce operators Q^{pq} by using the definition

$$Q \stackrel{\text{def}}{=} Q^{pq}\bar{\partial}_p\partial_q \tag{12.30a}$$

and expand these operators into the $\hbar$-power series

$$Q^{pq} = Q_0^{pq} + \hbar Q_1^{pq} + \hbar^2 Q_2^{pq} + \dots . \tag{12.31}$$

Here the leading term is the second-order differential operator

$$\begin{aligned} Q_0^{pq} &= (\bar{\partial}_r g^{ps})(\partial_s g^{rq}) - \Delta(g^{pq}) \\ &\quad - g^{rs}(\partial_s g^{pq})\bar{\partial}_r - g^{rs}(\bar{\partial}_r g^{pq})\partial_s - \tfrac{1}{2}g^{pq}\Delta. \end{aligned} \tag{12.32}$$

All other terms in (12.31) can also be calculated explicitly via the quantum metric g by using formulas (12.29), (12.30), and (12.21).

On Ω let us introduce the Liouville measure generated by the form ω (12.2), namely,

$$dm^\omega \stackrel{\text{def}}{=} \det|\omega|\, d\bar{z}\, dz$$

and express the reproducing measure dm via the Liouville measure:

$$dm = e^{\hbar f}\, dm^\omega. \tag{12.33}$$

We call f the "reproducing function." It is globally defined over Ω.

Theorem 12.3. *The reproducing function f in* (12.33) *is a solution of the differential equation*

$$df = \sum_{p,q=1}^{d} Q^{pq}\, d\omega_{pq}. \tag{12.34}$$

Here ω_{pq} are coefficients (12.10) *of the Kähler form ω and Q^{pq} are (pseudo)differential operators defined by* (12.30), (12.30a).

The operators in expansion (12.31) *determine closed* 1*-forms over the Kähler manifold Ω:*

$$\theta_m \stackrel{\text{def}}{=} \sum_{p,q=1}^{d} Q_m^{pq}\, d\omega_{pq}, \qquad m = 0, 1, 2, \dots,$$

so that equation (12.34) *reads*

$$df = \sum_{m=0}^{\infty} \hbar^m \theta_m. \tag{12.34a}$$

If Ω is simply connected, then (12.34a) *is globally solvable. In general, the cohomology classes $[\theta_m] \in H^1(\Omega, \mathbb{R})$ are the obstructions to the existence of the reproducing measure on Ω (smoothly depending on $\hbar$). The leading class $[\theta_0]$ is given by the* 1*-form on Ω:*

$$\theta_0 = \sum_{p,q} Q_0^{pq}\, d\omega_{pq},$$

where the second-order differential operators Q_0^{pq} are determined by (12.32).

Remark 12.2. We stress that all formal expansions (12.18), (12.19), (12.22), (12.31), (12.34a) are not purely $\hbar$-power series; their coefficients implicitly depend on $\hbar$, since the quantum metric g and the quantum symplectic form ω in general can depend on $\hbar$ in a very complicated way. For instance, if the Kähler manifold Ω is compact or just posesses a nontrivial 2-cycle, then the form ω has to satisfy the quantization condition $(1/2\pi\hbar)[\omega] \in H^2(\Omega, \mathbb{Z})$ which explicitly includes $\hbar$.

Remark 12.3. Equation (12.34a) determines the reproducing function f up to an arbitrary additive constant only. This constant can readily be obtained from the "normalization" condition (12.26). Note also that in the compact case the first relation (12.26) implies a formula for the dimension of the quantum function space $\mathcal{P}_a$. Thus we obtain

$$\frac{1}{(2\pi\hbar)^d}\int_\Omega e^{-F_a/\hbar}e^{\hbar f}\,dm^\omega = 1, \qquad \frac{1}{(2\pi\hbar)^d}\int_\Omega e^{\hbar f}\,dm^\omega = \dim \mathcal{P}_a.$$

13. Quantum restriction onto irreducible leaves

Recall that to each function $f \in \mathcal{F}_\hbar(\mathcal{M})$ we assigned the operator $L_{f,a}$ acting in the space $\mathcal{P}_a$ of antiholomorphic distributions. The algebra of such operators is realized as the algebra $\mathcal{F}_\hbar^a$ of the Wick symbols with product $\underset{a}{*}$ (12.1), (12.9).

Denote by f_a the Wick symbol of the operator $L_{f,a}$, i.e.,

$$L_{f,a} = \widehat{f_a} \equiv f_a(\overset{2}{\bar{z}}, \overset{1}{\hat{z}}), \tag{13.1}$$

$$f_a = \frac{1}{\mathcal{K}_a} L_{f,a}(\mathcal{K}_a). \tag{13.2}$$

We obtain the mapping

$$f \mapsto f_a, \qquad \mathcal{F}_\hbar(\mathcal{M}) \to \mathcal{F}_\hbar^a. \tag{13.3}$$

The results of Section 12 and Lemma 11.2 imply the statement.

Lemma 13.1. *For generic $a \in \mathcal{M}_0$, for which the matrix $\{\overline{C}_j, C_k\}_\mathcal{M}$ is not degenerate at a, the mapping* (13.3) *is a homomorphism of the normal $*$-product algebra $\mathcal{F}_\hbar(\mathcal{M})$* (*see Section* 10) *onto the Wick $\underset{a}{*}$-product algebra $\mathcal{F}_\hbar^a$* (12.1), (12.9). *The algebra $\mathcal{F}_\hbar^a$ is minimal, that is, its center is trivial. Moreover, the operators of the left and right multiplication in $\mathcal{F}_\hbar^a$ are given by*

$$f_a \underset{a}{*} \varphi = \frac{1}{\mathcal{K}_a} L_{f,a}(\mathcal{K}_a\varphi), \tag{13.4}$$

$$\varphi \underset{a}{*} f_a = \frac{1}{\mathcal{K}_a} R_{f,a}(\mathcal{K}_a\varphi). \tag{13.5}$$

Lemma 13.1 has the following simple consequence.

Lemma 13.2. (i) *Each function f_a* (13.2) *can be performed as*

$$f_a = f(\overset{3}{B}_a, \overset{2}{A}_a, \overset{1}{C}_a), \tag{13.6}$$

where the elements $B_a, A_a, C_a \in \mathcal{F}_\hbar^a$ on the right are obtained from the coordinate functions B, A, C by the procedure (13.2), *and the function of noncommuting elements in* (13.6) *is understood in the sense of the Wick star-product.*

(ii) *Let $M = M(B, A, C)$ be a Casimir element of the normal star-product algebra of functions over $\mathcal{M}$, that is, $M * f = f * M$ for all $f \in \mathcal{F}_\hbar(\mathcal{M})$. Then*

$$M_a = M(\overset{3}{B}_a, \overset{2}{A}_a, \overset{1}{C}_a) = M(0, a, 0). \tag{13.7}$$

(iii) *Let $A(t)$, $B(t)$, $C(t)$ be functions obtained from A, B, C by the shift per time t along the quantum Liouville–Heisenberg flow of type* (4.4). *Then for any Casimir function M, the following identity holds*:

$$M(\overset{3}{B}(t)_a, \overset{2}{A}(t)_a, \overset{1}{C}(t)_a) = M(0, a, 0).$$

Proof. Formula (13.6) follows from the statement of Lemma 13.1 that the mapping (13.3) is a homomorphism. Formula (13.7) follows from the minimality of $\mathcal{F}_\hbar^a$ (since M_a commutes with all functions, it must be a constant). □

We call the functions A_a, B_a, C_a *quantum normal coordinates*. For simplicity, we assume that these coordinates are global on $\mathcal{M}$.

Now let us define submanifolds $\Omega_a \subset \mathcal{M}$ by means of quantum normal coordinates as follows:

$$\Omega_a = \left\{(B, A, C) \mid B = \overline{C} = B_a(\bar{z}, z), \ A = A_a(\bar{z}, z)\right\}. \tag{13.8}$$

Then equation (13.7) reads

$$M\big(\overset{3}{B}\big|_{\Omega_a}, \overset{2}{A}\big|_{\Omega_a}, \overset{1}{C}\big|_{\Omega_a}\big) = \text{const}$$

for any Casimir function M. We call Ω_a the *quantum irreducible leaves*.

The algebra $\mathcal{F}_\hbar^a$ we identify with the algebra of functions over Ω_a, that is, $\mathcal{F}_\hbar^a \equiv \mathcal{F}_\hbar(\Omega_a)$. Let us denote $f_a = f\big|_{\widehat{\Omega_a}}$. The epimorphism (13.3)

$$f \mapsto f\big|_{\widehat{\Omega_a}}, \qquad \mathcal{F}_\hbar(\mathcal{M}) \to \mathcal{F}_\hbar(\Omega_a)$$

can be called the *quantum restriction* from $\mathcal{M}$ onto Ω_a. From the previous lemmas we obtain the theorem.

Theorem 13.1. *The quantum restriction onto irreducible leaves Ω_a (where a is a generic point on $\mathcal{M}_0$) possesses the following properties.*

(i) *The quantum restriction of the normal coordinate functions coincides with the classical restriction and is calculated via the quantum Kählerian potential by the formulas*:

$$\begin{aligned} B\big|_{\widehat{\Omega_a}} &= B\big|_{\Omega_a} = \partial F_a(\bar{z}, z), \\ C\big|_{\widehat{\Omega_a}} &= C\big|_{\Omega_a} = \bar{\partial} F_a(\bar{z}, z), \\ A\big|_{\widehat{\Omega_a}} &= A\big|_{\Omega_a} = \mathcal{L}_{A,a}\big(\partial F_a \overset{1}{+} \hbar\partial, \overset{2}{z}\big)\, 1. \end{aligned} \tag{13.8a}$$

Here the quantum Kählerian potential F_a satisfies problem (11.17), (11.17a), *and $\mathcal{L}_{C,a}$ and $\mathcal{L}_{A,a}$ are symbols of operators of the left regular representation $C*$, $A*$ restricted onto the annihilation subspace $E_a \subset \mathcal{E}^{\mathbb{C}}$.*

(ii) *General formula for the quantum restriction is the following*:

$$\begin{aligned} f\big|_{\widehat{\Omega_a}} &= f\big(\overset{3}{B}\big|_{\widehat{\Omega_a}}, \overset{2}{A}\big|_{\widehat{\Omega_a}}, \overset{1}{C}\big|_{\widehat{\Omega_a}}\big) \\ &= \overline{\mathcal{L}}^a_{\bar{f}}(\overset{1}{\bar{\partial} F_a + \hbar\bar{\partial}}, \overset{2}{\bar{z}})\, 1 = \mathcal{L}_{f,a}(\overset{1}{\partial F_a + \hbar\partial}, \overset{2}{z})\, 1 \\ &= \mathcal{R}^a_f(\overset{1}{\bar{\partial} F_a + \hbar\bar{\partial}}, \overset{2}{\bar{z}})\, 1 = \overline{\mathcal{R}}_{\bar{f},a}(\overset{1}{\partial F_a + \hbar\partial}, \overset{2}{z})\, 1. \end{aligned}$$

(iii) *The quantum restriction is a homomorphism* $\mathcal{F}_\hbar(\mathcal{M}) \to \mathcal{F}_\hbar(\Omega_a)$, *i.e.*,

$$(f\big|_{\widehat{\Omega_a}}) \underset{a}{*} (g\big|_{\widehat{\Omega_a}}) = (f * g)\big|_{\widehat{\Omega_a}}.$$

(iv) *For any Casimir function* $M \in \mathcal{F}_\hbar(\mathcal{M})$, *we have*

$$M\big|_{\widehat{\Omega_a}} = M\big(\overset{3}{B}\big|_{\widehat{\Omega_a}}, \overset{2}{A}\big|_{\widehat{\Omega_a}}, \overset{1}{C}\big|_{\widehat{\Omega_a}}\big) = \text{const} = M(0, a, 0).$$

(v) *After the shift along any quantum Liouville–Heisenberg flow the functions* $B(t)\big|_{\widehat{\Omega_a}}$, $A(t)\big|_{\widehat{\Omega_a}}$, $C(t)\big|_{\widehat{\Omega_a}}$ *stay on the same quantum level* (13.7), *i.e.*,

$$M\big(\overset{3}{B}(t)\big|_{\widehat{\Omega_a}}, \overset{2}{A}(t)\big|_{\widehat{\Omega_a}}, \overset{1}{C}(t)\big|_{\widehat{\Omega_a}}\big) = M(0, a, 0). \tag{13.9}$$

Remark 13.1. The quantum leaves Ω_a are transversal to the vacuum submanifold $\mathcal{M}_0$ almost for all "a" (while the matrix of Poisson brackets $\{\overline{C}_j, C_k\}$ is not degenerate on Ω_a; see the definition of the partial Kählerian structure above). Moreover, it is not difficult to prove that

$$a \in \Omega_a \cap \mathcal{M}_0$$

(but, may be, there are some other points in this intersection if the correspondence $a \to \Omega_a$ is not one-to-one). The intersection $\Omega_a \cap \mathcal{M}_0$ plays the role of the vacuum submanifold in Ω_a.

Also note that the quantum leaves are automatically endowed with a Kählerian structure by means of the form ω (12.2).

At last, we see from (13.9) that quantum leaves are orbits of the pseudogroup of quantum Liouville–Heisenberg transformations.

In Section 15 we consider the classical limit of the quantum restriction as $\hbar \to 0$.

Remark 13.2. In notation of Remark 11.4 we obtain

$$\hat{f}\big|_{\mathcal{H}_a} = \widehat{f\big|_{\widehat{\Omega_a}}}. \tag{13.10}$$

On the left the symbol f is quantized by the normal ordering procedure (11.11) and then the operator is restricted onto the irreducible component $\mathcal{H}_a \subset \mathcal{H}$. On the right the symbol f is first restricted (in the quantum sense) onto the irreducible leaf Ω_a, and then is quantized by the Wick procedure over Ω_a. By definition, to each function $\varphi = \varphi(\bar{z}, z)$ this last procedure assigns an operator $\hat{\varphi}$ in $\mathcal{H}_a$ determined by equality (12.24). The relation (13.10) precisely represents the statement: "quantization commutes with restriction onto irreducible leaves."

14. Quantum restriction and quantum reduction

Let us now consider and compare reduction homomorphisms over $\mathcal{M}$ and over its irreducible leaves Ω_a.

The construction of the quantum phase space over $\mathcal{M}$ was explained in Sections 9 and 10. Here we consider only the normal ordering complex version of Section 10; so, the manifold $\mathcal{M}$ and its phase space $\mathcal{E}$ are partially complexified. The corresponding reduction homomorphisms are given by the mappings

$$\begin{aligned} &\mathcal{F}_\hbar(\mathcal{M}^{\mathbb{C}}) \xrightarrow{\hat{\ell}^*} \mathcal{F}_\hbar(\mathcal{E}^{\mathbb{C}}), \qquad && \hat{\ell}^* : f(B, A, C) \to \mathcal{L}_f(B, z; A, x; C, \bar{z}),\\ &\mathcal{F}_\hbar(\mathcal{M}^{\mathbb{C}})^{(-)} \xrightarrow{\hat{r}^*} \mathcal{F}_\hbar(\mathcal{E}^{\mathbb{C}}), \qquad && \hat{r}^* : f(B, A, C) \to \mathcal{R}_f(B, z; A, x; C, \bar{z}), \end{aligned} \tag{14.1}$$

where $\mathcal{L}_f$ and $\mathcal{R}_f$ are the symbols of operators of the left and right regular representations (see Theorem 10.1). The phase space $\mathcal{E}^{\mathbb{C}}$ is a neighborhood of the zero section in $T\mathcal{M}^{\mathbb{C}}$, the quantum structure over $\mathcal{E}^{\mathbb{C}}$ is given by the normal ordering rule

$$z^s * B_\ell = z^s B_\ell + \hbar\delta^s_\ell, \qquad \bar{z}^s * C_\ell = \bar{z}^s C_\ell + \hbar\delta^s_\ell, \qquad x^\mu * A_\nu = x^\mu A_\nu + \hbar\delta^\mu_\nu$$

(all other pairwise quantum products of coordinate functions are trivial, i.e., coincide with the ordinary commutative products). The classical symplectic form on $\mathcal{E}^{\mathbb{C}}$ corresponding to this quantum structure is the following:

$$\omega_{\mathcal{E}^{\mathbb{C}}} = i\big(dB \wedge dz + dA \wedge dx + dC \wedge d\bar{z}\big).$$

The manifolds $\mathcal{M}^{\mathbb{C}}$ is embedded into $\mathcal{E}^{\mathbb{C}}$ as the Lagrangian zero section $\{z = \bar{z} = x = 0\}$, and the embedding $i_{\mathcal{M}^{\mathbb{C}}} : \mathcal{M}^{\mathbb{C}} \hookrightarrow \mathcal{E}^{\mathbb{C}}$ is consistent with homomorphisms (14.1):

$$i^*_{\mathcal{M}^{\mathbb{C}}} \circ \hat{\ell}^* = i^*_{\mathcal{M}^{\mathbb{C}}} \circ \hat{r}^* = I.$$

Now we need a phase space and reduction homomorphisms over the irreducible leaves $\Omega_a \subset \mathcal{M}$. For each $\varphi \in \mathcal{F}_\hbar(\Omega_a)$, let us represent the left multiplication on φ in the form of the Wick operator with some symbol, which we denote by $\hat{\ell}^*\varphi$. Thus, by definition,

$$\varphi \underset{a}{*} \stackrel{\text{def}}{=} (\hat{\ell}^*\varphi)(\overset{2}{\bar{z}}, \overset{2}{z}, \overset{1}{\hat{\hat{z}}}, \overset{1}{\hat{z}}). \tag{14.2}$$

Here $\hat{z}$ is the antiholomorphic operator (acting by $\bar{z}$) defined by (12.4). The operator $\hat{\hat{z}}$ is holomorphic, acts by the variable z and is defined by the relation conjugate to (12.4).

From the third property in (12.3) we know that the symbol $(\hat{\ell}^*_a\varphi)(\bar{z}, z; \overline{w}, w)$ does not actually depend on $\overline{w}$; and from (12.8) it follows that

$$(\hat{\ell}^*_a\varphi)(\bar{z}, z; \overline{w}, w) = \frac{1}{\mathcal{K}_a(\bar{z}, z)\mathcal{K}_a(\bar{z}, w)}\varphi(\overset{2}{\bar{z}}, \overset{1}{\hat{z}})(\mathcal{K}_a(\bar{z}, z)\mathcal{K}_a(\bar{z}, w)). \tag{14.3}$$

The right multiplication operator is represented in the same way

$$\underset{a}{*}\, \varphi \stackrel{\text{def}}{=} (\hat{r}^*\varphi)(\overset{2}{\bar{z}}, \overset{2}{z}, \overset{1}{\hat{\hat{z}}}, \overset{1}{\hat{z}}), \tag{14.2a}$$

and the symbol $\hat{r}^*\varphi$ can be obtained from $\hat{\ell}^*\varphi$ just by the complex conjugation

$$(\hat{r}^*_a\varphi)(\bar{z}, z; \overline{w}, w) = \overline{(\hat{\ell}^*_a\overline{\varphi})(\bar{z}, z; \overline{w}, w)}.$$

Thus, we have two reduction homomorphisms

$$\hat{\ell}_a^*\colon \mathcal{F}_\hbar(\Omega_a^{\mathbb{C}}) \to \mathcal{F}_\hbar(\mathcal{E}_a^{\mathbb{C}}), \qquad \hat{r}_a^*\colon \mathcal{F}_\hbar(\Omega_a^{\mathbb{C}})^{(-)} \to \mathcal{F}_\hbar(\mathcal{E}_a^{\mathbb{C}}). \tag{14.4}$$

Here the phase space $\mathcal{E}_a^{\mathbb{C}} \approx \Omega_a^{\mathbb{C}} \times \Omega_a^{\mathbb{C}}$ is endowed with the Wick product

$$w^r * \bar{z}^\ell = w^r \bar{z}^\ell + \hbar N^{\ell r}(\bar{z}, w) \tag{14.5}$$

(all other pairwise products of coordinate functions z, $\bar{z}$, w, $\overline{w}$ over $\mathcal{E}_a^{\mathbb{C}}$ are either trivial or given by complex conjugation of (14.5)). Recall that the quantum tensor $N^{\ell r}$ in (14.5) is obtained from (12.17a); its leading $\hbar$-expansion terms were demonstrated in (12.18). The symplectic form on $\mathcal{E}_a^{\mathbb{C}}$ corresponding to the quantum product (14.5) is

$$\omega_{\mathcal{E}_a^{\mathbb{C}}} = i\bar{\partial}_s\partial_\ell F_a(\bar{z}, w)\, d\bar{z}^s \wedge dw^\ell + i\bar{\partial}_s\partial_\ell F_a(\overline{w}, z)\, d\overline{w}^s \wedge dz^\ell. \tag{14.6}$$

The real phase space $\mathcal{E}_a \approx \Omega_a \times \Omega_a$ is embedded into $\mathcal{E}_a^{\mathbb{C}}$ as the graph of the complex involution. The manifold $\Omega_a^{\mathbb{C}}$ is embedded into its phase space $\mathcal{E}_a^{\mathbb{C}}$ as the Lagrangian zero section $\{w = \overline{w} = 0\}$. This embedding $i_{\Omega_a^{\mathbb{C}}}\colon \Omega_a^{\mathbb{C}} \hookrightarrow \mathcal{E}_a^{\mathbb{C}}$ is consistent with the reduction homomorphisms (14.4), that is,

$$i^*_{\Omega_a^{\mathbb{C}}} \circ \hat{\ell}_a^* = i^*_{\Omega_a^{\mathbb{C}}} \circ \hat{r}_a^* = I.$$

Now we recall that in the phase space $\mathcal{E}^{\mathbb{C}}$ over $\mathcal{M}^{\mathbb{C}}$ we have the creation E_a and the annihilation E^a submanifolds (see in Section 11). We can define two quantum embeddings of $\mathcal{E}_a^{\mathbb{C}}$ into $\mathcal{E}^{\mathbb{C}}$. First we identify $\mathcal{E}_a^{\mathbb{C}}$ with $E_a \subset \mathcal{E}^{\mathbb{C}}$ by the mapping

$$(\bar{z}, z; \overline{w}, w) \to \big(\partial F_a(\bar{z}, z) + \partial F_a(\overline{w}, z), z; a, 0; 0, 0\big). \tag{14.7}$$

For each function $k = k(B, z; A, x; C, \bar{z})$ on $\mathcal{E}^{\mathbb{C}}$, we define its quantum restriction onto $E_a \approx \mathcal{E}_a^{\mathbb{C}}$ as follows:

$$k\big|_{\widehat{E_a}} = k\big(\partial F_a(\bar{z}, z) + \partial F_a(\overline{w}, z) \overset{1}{+} \hbar\partial/\partial z, \overset{2}{z}; a, 0; 0, 0\big)1. \tag{14.7a}$$

In the same way, we can identify $\mathcal{E}_a^{\mathbb{C}}$ with $E^a \subset \mathcal{E}^{\mathbb{C}}$ by the mapping

$$(\bar{z}, z; \overline{w}, w) \to \big(0, 0; a, 0; \bar{\partial} F_a(\bar{z}, z) + \bar{\partial} F_a(\bar{z}, w), \bar{z}\big). \tag{14.8}$$

Then for each function k on $\mathcal{E}^{\mathbb{C}}$ we define its quantum restriction onto $E^a \approx \mathcal{E}_a^{\mathbb{C}}$ as follows:

$$k\big|_{\widehat{E^a}} = k\big(0, 0; a, 0; \bar{\partial} F_a(\bar{z}, z) + \bar{\partial} F_a(\bar{z}, w) \overset{1}{+} \hbar\partial/\partial\bar{z}, \overset{2}{\bar{z}}\big)1. \tag{14.8a}$$

The preparation is over, now we can formulate the final statement which connects the quantum reduction homomorphisms over $\mathcal{M}$ and over Ω_a with the procedure of quantum restriction.

First of all, note that formulas (13.4) and (13.5) read

$$\begin{aligned} (f\big|_{\widehat{\Omega_a}}) \underset{a}{*} &= \mathcal{R}_f^a\big(\bar{\partial} F_a \overset{1}{+} \hbar\bar{\partial}, \overset{2}{\bar{z}}\big) = \overline{\mathcal{L}}_{\bar{f}}^a\big(\bar{\partial} F_a \overset{1}{+} \hbar\bar{\partial}, \overset{2}{\bar{z}}\big), \\ \underset{a}{*} (f\big|_{\widehat{\Omega_a}}) &= \mathcal{L}_{f,a}\big(\partial F_a \overset{1}{+} \hbar\partial, \overset{2}{z}\big) = \overline{\mathcal{R}}_{\bar{f},a}\big(\partial F_a \overset{1}{+} \hbar\partial, \overset{2}{z}\big). \end{aligned} \tag{14.9}$$

Let us calculate the Wick symbols of all operators in these equalities. Say, on the first line of (14.9), the Wick symbol of the operator $(f\big|_{\widehat{\Omega_a}}) \underset{a}{*}$, by definition, is equal to $\hat{\ell}_a^*(f\big|_{\widehat{\Omega_a}})$. On the other hand, the Wick symbol of the operator

$$\mathcal{R}_f^a\big(\overset{1}{\bar\partial F_a + \hbar\bar\partial}, \overset{2}{\bar z}\big)$$

is equal to

$$\mathcal{R}_f^a\big(\bar\partial F_a(\bar z, z) + \bar\partial F_a(\bar z, w) \overset{1}{+} \hbar\partial/\partial\bar z, \overset{2}{\bar z}\big)\, 1,$$

and so, in view of (14.8a), it coincides with the quantum restriction $\mathcal{R}_f\big|_{\widehat{E^a}}$. If we recall that $\mathcal{R}_f = \hat r^*(f)$, we see that this last symbol is equal to $\hat r^*(f)\big|_{\widehat{E^a}}$. Thus, we have proved the following theorem.

Theorem 14.1. *For each generic $a \in \mathcal{M}_0$, the quantum reduction homomorphisms over the quantum manifold $\mathcal{M}$ and over its irreducible leaf Ω_a are related to each other by the quantum restriction mappings*:

$$\hat\ell_a^*\big(f\big|_{\widehat{\Omega_a}}\big) = \hat r^*(f)\big|_{\widehat{E^a}}, \qquad \hat r_a^*\big(f\big|_{\widehat{\Omega_a}}\big) = \hat\ell_a^*(f)\big|_{\widehat{E_a}}. \tag{14.10}$$

Here the annihilation submanifold E^a or the creation submanifold E_a are identified with the phase space (*symplectic groupoid*) *$\mathcal{E}_a^{\mathbb{C}}$ over $\Omega_a^{\mathbb{C}}$ by means of embeddings* (14.8) *or* (14.7).

This is exactly the statement announced in the Introduction: "quantum reduction homomorphisms commute with quantum restriction" with the essential remark: left and right reductions change their places in this process. Another form of relations (14.10) is the following:

$$\hat\ell_a^*\big(f\big|_{\widehat{\Omega_a}}\big) = \bar{\hat\ell}^*(f)\big|_{\widehat{E^a}}, \qquad \hat r_a^*\big(f\big|_{\widehat{\Omega_a}}\big) = \bar{\hat r}_a^*(f)\big|_{\widehat{E_a}}. \tag{14.10a}$$

Here the left and right are not transposed, but an additional complex conjugation appears.

15. Classical geometric consequences of quantum calculations

In this last section we look at quantum results obtained in Sections 10–14 in the classical limit $\hbar = 0$.

First let us describe explicitly the reduction mappings (10.3).

Let the Poisson brackets on $\mathcal{M}$ consistent with a partial complex structure be given by the relations

$$\begin{aligned}
&\{C_s, B_\ell\}_{\mathcal{M}} = \Psi_{s\ell}(B, A, C),\\
&\{A^\mu, B_s\}_{\mathcal{M}} = \Gamma_s^\mu(B, A),\\
&\{A^\mu, A^\nu\}_{\mathcal{M}} = \{B_s, B_\ell\}_{\mathcal{M}} = \{C_s, C_\ell\}_{\mathcal{M}} = 0.
\end{aligned} \tag{15.1}$$

Denote by $\gamma^z(B, A)$ the shift of the point $A \in \mathbb{R}^k$ per unit time along trajectories of the vector field $z^s\Gamma_s^\mu(B, A)\,\partial/\partial A^\mu$. Denote by $\sigma^x(A, C)$ the shift of the point $C \in \mathbb{C}^d$ per unit time along trajectories of the vector field $x_\mu\bar\Gamma_\ell^\mu(C, A)\,\partial/\partial C_\ell$. At last, denote by $\lambda^z(B, A, C)$ the shift of the point $C \in \mathbb{C}^d$ per time $t = 1$ along trajectories of the time-dependent vector field $z^s\Psi_{s\ell}\big(B, \gamma^{tz}(B, A), C\big)\,\partial/\partial C_\ell$.

Theorem 15.1. *Suppose that the Poisson brackets on $\mathcal{M}$ are determined by relations* (15.1). *Then the left reduction mapping* (10.3) $\ell^{\mathbb{C}} = (\mathcal{L}_B^{(0)}, \mathcal{L}_A^{(0)}, \mathcal{L}_C^{(0)})$ *is given by the following formulas*:

$$
\begin{aligned}
&\mathcal{L}_B^{(0)}(B, z; A, x; C, \bar{z}) = B,\\
&\mathcal{L}_A^{(0)}(B, z; A, x; C, \bar{z}) = \gamma^z(B, A),\\
&\mathcal{L}_C^{(0)}(B, z; A, x; C, \bar{z}) = \sigma^x\big(\gamma^z(B, A), \lambda^z(B, A, C)\big).
\end{aligned}
$$

The right reduction mapping (10.3) *is given by* $r^{\mathbb{C}} = (\overline{\mathcal{L}}_C^{(0)}, \overline{\mathcal{L}}_A^{(0)}, \overline{\mathcal{L}}_B^{(0)})$. *The mappings* $\ell^{\mathbb{C}}$ *and* $r^{\mathbb{C}}$ *mutually commute; they are Poisson and anti-Poisson mappings* $\mathcal{E}^{\mathbb{C}} \to \mathcal{M}^{\mathbb{C}}$. *The quantum homomorphisms* (14.1) *in the classical limit are determined by these reduction mappings*:

$$
\hat{\ell}^*(f)\big|_{\hbar=0} = f \circ \ell^{\mathbb{C}}, \qquad \hat{r}^*(f)\big|_{\hbar=0} = f \circ r^{\mathbb{C}}.
$$

This statement is the complex analog of the statement obtained in [42] (see also [52]) for the case of real Poisson brackets.

Now, since we know the leading terms of the functions $\mathcal{L}_C$ and $\mathcal{L}_A$ explicitly, we can derive the equations for the leading part $F_a^{(0)}$ of the Kählerian potential from (11.17), (11.17a), and (11.18).

Theorem 15.2. *The leading part of the potential of the Kählerian form* (12.2a) *is the solution of the following Cauchy problem for the Hamilton–Jacobi equation*:

$$
\frac{\partial F_a^{(0)}}{\partial \bar{z}} = \lambda^z\bigg(\frac{\partial F_a^{(0)}}{\partial z}, a, 0\bigg), \qquad F_a^{(0)}\big|_{\bar{z}=0} = 0, \tag{15.2}
$$

with the additional condition

$$
\operatorname{Im} \gamma^z\bigg(\frac{\partial F_a^{(0)}}{\partial z}, a\bigg) = 0. \tag{15.3}
$$

Now the classical limit of the quantum irreducible leaves (13.8) can be derived from (13.8) and (13.8a).

Theorem 15.3. *For any point $a \in \mathcal{M}_0$, at which the matrix Ψ is not degenerate, the symplectic leaf in $\mathcal{M}$, passing through the point a, is given by the equations*

$$
\Omega_a^{(0)} = \big\{A = \gamma^z\big(\partial F_a^{(0)}(\bar{z}, z), a\big), B = \partial F_a^{(0)}(\bar{z}, z) = \overline{C}\big\}, \tag{15.4}
$$

where $F_a^{(0)}$ is the solution of the problem (14.2), (14.3). *Formulas* (15.4) *determine the complex structure on each leaf $\Omega_a^{(0)} \subset \mathcal{M}$ with complex coordinates z. The Kirillov form on the leaf* (15.4) (*see general definition in* [56]) *is given by formula* (12.2a). *For $\hbar = 0$ the quantum restriction onto Ω_a coincides with the classical restriction onto the symplectic leaf $\Omega_a^{(0)}$.*

Note that Theorem 15.3 presents a constructive way for calculating symplectic leaves for Poisson brackets (15.1), as well as the complex structure on them and the Kirillov form. This is a purely classical geometric result, although it was obtained from quantum calculations.

The result of Theorem 15.3 can be also performed as follows.

Corollary 15.1. *The mapping*

$$\gamma_a\colon (\bar z, z) \mapsto \big(\partial F_a^{(0)}(\bar z, z), z; a, 0; 0, 0\big) \tag{15.5}$$

is an embedding of $\Omega_a^{(0)}$ into the creation submanifold $E_a \subset \mathcal{E}^{\mathbb{C}}$ such that

$$\ell^{\mathbb{C}} \circ \gamma_a = \mathrm{id}_{\Omega_a^{(0)}}.$$

The "conjugate" mapping

$$\gamma^a\colon (\bar z, z) \mapsto \big(0, 0; a, 0; \bar\partial F_a^{(0)}(\bar z, z), \bar z\big) \tag{15.5a}$$

is an embedding of $\Omega_a^{(0)}$ into the annihilation submanifold $E^a \subset \mathcal{E}^{\mathbb{C}}$ such that

$$r^{\mathbb{C}} \circ \gamma^a = \mathrm{id}_{\Omega_a^{(0)}}.$$

Now let us look at the classical limit of the reduction homomorphisms (14.4) over the leaf Ω_a. From (14.3) it follows that we need to calculate the operator

$$\frac{1}{\mathcal{K}_a(\bar z, z)\mathcal{K}_a(\bar z, w)} \circ \hat z \circ \mathcal{K}_a(\bar z, z)\mathcal{K}_a(\bar z, w) \overset{\text{def}}{=} \rho_\hbar(z, w).$$

From equation (12.9a) we have

$$\bar\partial F_a\big(\overset{2}{\bar z}, \rho_\hbar(\overset{1}{z}, w)\big) = \hbar\partial/\partial\bar z + \bar\partial F_a(\bar z, z) + \bar\partial F_a(\bar z, w).$$

Hence,

$$\rho_\hbar(z, w) = (\bar\partial F_a)^{-1}\big(\overset{2}{\bar z}, \hbar\partial/\partial\bar z \overset{1}{+} \bar\partial F_a(\bar z, z) + \bar\partial F_a(\bar z, w)\big) + O(\hbar), \tag{15.6}$$

where we denote by $(\bar\partial F_a)^{-1}$ the inverse mapping, i.e.,

$$\bar\partial F_a(\bar z, \rho) = \theta \;\Rightarrow\; \rho \overset{\text{def}}{=} (\bar\partial F_a)^{-1}(\bar z, \theta).$$

Thus, at $\hbar = 0$, we obtain from (15.6):

$$\rho_0(z, w) = (\bar\partial F_a^{(0)})^{-1}\big(\bar z, \bar\partial F_a^{(0)}(\bar z, z) + \bar\partial F_a^{(0)}(\bar z, w)\big).$$

After the substitution into (14.3), we derive the following formulas for the reduction homomorphisms.

Theorem 15.4. *In the classical limit the reduction homomorphisms* (14.4) *are given by*

$$\hat\ell_a^*(f)\big|_{\hbar=0} = f \circ \ell_a^{\mathbb{C}}, \qquad \hat r_a^*(f)\big|_{\hbar=0} = f \circ r_a^{\mathbb{C}},$$

where $\ell_a^{\mathbb{C}}$ and $r_a^{\mathbb{C}}$ are Poisson mappings

$$\ell_a^{\mathbb{C}}\colon \mathcal{E}_a^{\mathbb{C}} \to \Omega_a^{(0)\mathbb{C}}, \qquad r_a^{\mathbb{C}}\colon \mathcal{E}_a^{\mathbb{C}} \to \Omega_a^{(0)\mathbb{C}(-)},$$

such that

$$\{\ell_a^{\mathbb{C}}, r_a^{\mathbb{C}}\}_{\mathcal{E}_a^{\mathbb{C}}} = 0, \qquad \ell_a^{\mathbb{C}}\big|_{\mathcal{M}^{\mathbb{C}}} = r_a^{\mathbb{C}}\big|_{\mathcal{M}^{\mathbb{C}}} = \mathrm{id}.$$

The explicit formulas for these mappings are

$$\ell_a^{\mathbb{C}}(\bar{z}, z; \bar{w}, w) = \big(\bar{z}, (\bar{\partial} F_a^{(0)})^{-1}\big(\bar{z}, \bar{\partial} F_a^{(0)}(\bar{z}, z) + \bar{\partial} F_a^{(0)}(\bar{z}, w)\big)\big),$$
$$r_a^{\mathbb{C}}(\bar{z}, z; \bar{w}, w) = \big((\partial F_a^{(0)})^{-1}\big(\partial F_a^{(0)}(\bar{z}, z) + \partial F_a^{(0)}(\bar{w}, z), z\big), z\big),$$

where $F_a^{(0)}$ *is the solution of problem* (14.2), (14.3).

This theorem describes the classical phase space over the symplectic leaf $\Omega_a^{(0)} \subset \mathcal{M}$ in terms of the Kählerian potential $F_a^{(0)}$ of the Kirillov form (12.2a).

The classical limit of our last quantum Theorem 14.1 is now clear, and we leave its formulation to the reader.

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